UNIVERSIDADE da MADEIRA

# Continuum Mechanics

**Series of exercises 1** - Indicial notation

Note: Exercises marked with ✠ will be solved in classes.

1. ✠ Given

$$[S_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix}$$

Evaluate

$$S_{ii},\ S_{ij}S_{ij},\ S_{jk}S_{jk},\ S_{mn}S_{nm}$$

Solution:

$$S_{ii} = 5,\ S_{ij}S_{ij} = 28,\ S_{jk}S_{jk} = 28,\ S_{mn}S_{nm} = 23.$$

2. Write the full form of the expression

$$T_{ij} = A_{im}A_{jm}$$

Solution:

$$\begin{aligned} T_{11} &= A_{1m}A_{1m} = A_{11}A_{11} + A_{12}A_{12} + A_{13}A_{13} \\ T_{12} &= A_{1m}A_{2m} = A_{11}A_{21} + A_{12}A_{22} + A_{13}A_{23} \\ &\ldots \\ T_{33} &= A_{3m}A_{3m} = A_{31}A_{31} + A_{32}A_{32} + A_{33}A_{33} \end{aligned}$$

3. ✠ Determine which of the following equations have the same meaning as $a_i = Q_{ij}a'_j$

$$a_l = Q_{lm}a'_m$$

$$a_p = Q_{qp}a'_q$$

$$a_m = a'_n Q_{mn}$$

Solution: $a_l = Q_{lm}a'_m$ (yes), $a_p = Q_{qp}a'_q$ (no), $a_m = a'_n Q_{mn}$ (yes)

4. Consider the equation

$$a_i + b_j = 0$$

What can one say about the quantities $a_1$, $a_2$, $a_3$ e $b_1$, $b_2$, $b_3$?

Solution: $b_1 = b_2 = b_3 = -a_1 = -a_2 = -a_3$.

5. Given

$$\mathbf{a} = [a_i] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},\quad \left[\widehat{B}\right] = [B_{ij}] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix},\quad \left[\widehat{C}\right] = [C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Write the following expressions in indicial notation:

(a) $\widehat{D} = \underbrace{\widetilde{\widetilde{B}}}_{=\widehat{B}^T}$

(b) $\mathbf{b} = \widehat{B}\mathbf{a}$

(c) $\widehat{D} = \widehat{B}\widehat{C}$

(d) $\widehat{D} = \widehat{B}\widetilde{\widetilde{C}}$

Solution: a) $(D_{ij} = B_{ji})$; b) $(b_i = B_{ij}a_j)$; c) $(D_{ij} = B_{im}C_{mj})$; d) $(D_{ij} = B_{im}C_{jm})$

6. Write the complete form of the equation $a_i = U_{im}V_{mk}c_k$.

Solution:

$$a_i = U_{i1}\left(V_{11}c_1 + V_{12}c_2 + V_{13}c_3\right) + U_{i2}\left(V_{21}c_1 + V_{22}c_2 + V_{23}c_3\right) + U_{i3}\left(V_{31}c_1 + V_{32}c_2 + V_{33}c_3\right)$$

That is, our equation actually represents a system of three equations, each of which has the sum of nine terms.

7. Given

$$T_{ij} = 2\mu E_{ij} + \lambda E_{kk}\delta_{ij}$$

Find

$$W = \frac{1}{2}T_{ij}E_{ij}$$
$$p = T_{ij}T_{ij}$$

Solution:

$$W = \mu E_{ij}E_{ij} + \frac{\lambda}{2}\left(E_{kk}\right)^2$$
$$p = 4\mu^2 E_{ij}E_{ij} + \left(E_{kk}\right)^2\left(4\mu\lambda + 3\lambda^2\right)$$

8. ✠ Given

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \widehat{S} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix}$$

Find

$$\widehat{T},\ \text{with } T_{ij} = \varepsilon_{ijk}a_k$$
$$\mathbf{c},\ \text{with } c_i = \varepsilon_{ijk}S_{jk}$$
$$\mathbf{d},\ \text{with } d_k = \varepsilon_{ijk}a_i b_j$$

Solution:

$$\widehat{T} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$\mathbf{d} = \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix}.$$

9. ✠ Given

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \ \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

where

$$d_k = \varepsilon_{ijk} a_i b_j.$$

Prove that

$$\mathbf{d} = \mathbf{a} \times \mathbf{b}$$

10. ✠ Prove that conditions

$$\varepsilon_{ijk} T_{jk} = 0 \text{ and } T_{ij} = T_{ji}$$

are equivalent.

11. ✠ Show that

$$\delta_{ij} \varepsilon_{ijk} = 0$$

12. Write all contractions of $E_{ij} F_{km}$ such that the result is a quantity with two indices.

Solution:

$$E_{ij} F_{im} = G_{jm} \quad E_{ij} F_{ki} = H_{jk} \quad E_{ij} F_{jm} = Q_{im} \quad E_{ij} F_{kj} = R_{ik} \quad E_{ii} F_{km} = K_{km} \quad E_{ij} F_{kk} = P_{ij}$$

13. ✠ Prove that

$$\varepsilon_{ijm} \varepsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \tag{1}$$

14. ✠ By contraction of the formula:

$$\varepsilon_{ijm} \varepsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$$

show that

$$\varepsilon_{ilm} \varepsilon_{jlm} = 2\delta_{ij}$$

and determine $\varepsilon_{ijk} \varepsilon_{ijk}$.

Solution:

$$\varepsilon_{ijk} \varepsilon_{ijk} = 2\delta_{ii} = 6$$

15. Write the formula

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} \tag{2}$$

in indicial notation and prove the result directly.

Solution:

$$\varepsilon_{lkn} \varepsilon_{ijk} a_l b_i c_j = a_i c_i b_n - a_i b_i c_n$$

16. Prove that if

$$T_{ij} = -T_{ji},$$

then

$$T_{ij} a_i a_j = 0$$

17. Prove that if

$$T_{ij} = -T_{ji} \text{ e } S_{ij} = S_{ji},$$

then

$$T_{ij} S_{ij} = 0$$

18. ✠ Represent a matrix $S_{ij}$ as the sum of a symmetric matrix with an antisymmetric matrix.

$\boxed{\text{Solution:}}$

$$T_{ij} = \frac{S_{ij} + S_{ji}}{2}, \quad R_{ij} = \frac{S_{ij} - S_{ji}}{2}$$

19. ✠ Given a function $f(x_1, x_2, x_3)$, write the differential of this function in indicial notation.

$\boxed{\text{Solution:}}$

$$df = \frac{\partial f}{\partial x_i} dx_i$$

20. Prove the formula

$$\det[A_{ij}] = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$$