UNIVERSIDADE da MADEIRA

Continuum Mechanics

Series of exercises 3 - Kinematics of a continuum

1. Let the motion of a body be given in component form as

$$x_1 = X_1 + t^2 X_2; \quad x_2 = X_2 + t^2 X_1; \quad x_3 = X_3$$

   Determine

   (a) The path of the particle originally at $\mathbf{X} = (1, 2, 1)$

   (b) The velocity and acceleration components of the same particle when $t = 2\,\mathrm{s}$.

2. Invert the motion equations of the preceding exercise to obtain $X = X(x, t)$ and determine the velocity and acceleration components of the particle at $x(1, 0, 1)$ when $t = 2\,\mathrm{s}$.

3. Let the motion equations be given in component form by the Lagrangian description

$$x_1 = X_1 e^t + X_3 \left(e^t - 1\right), x_2 = X_3 \left(e^t - e^{-t}\right) + X_2, x_3 = X_3$$

   Determine the Eulerian description of this motion.

4. For the motion of the preceding exercise determine the velocity and acceleration fields, and express these in both Lagrangian and Eulerian forms.

5. The position at time t, of a particle initially at (X_1, X_2, X_3), is given by the equations:

$$x_1 = X_1 + (X_1 + X_2)\,t; \quad x_2 = X_2 + (X_1 + X_2)\,t; \quad x_3 = X_3$$

   (a) Find the velocity at $t = 2$ for the particle which was at $(1, 1, 0)$ at the reference time.

   (b) Find the velocity at $t = 2$ for the particle which is at the position $(1, 1, 0)$ at $t = 2$.

6. Let a certain motion of a continuum be given by the component equations,

$$x_1 = X_1 e^{-t}; \quad x_2 = X_2 e^t; \quad x_3 = X_3 + X_2 \left(e^{-t} - 1\right)$$

   and let the temperature field of the body be given by the spatial description,

$$\theta = e^{-t} (x_1 - 2x_2 + 3x_3)$$

   Determine the velocity field in spatial form, and using that, compute the material derivative $D\theta/Dt$ of the temperature field.

7. Given the motion of a continuum to be

$$x_1 = X_1 + ktX_2; \quad x_2 = X_2; \quad x_3 = X_3$$

   If the temperature field is given by the spatial description

$$\theta = x_1 + x_2$$

   (a) find the material description of temperature.

   (b) obtain the velocity and rate of change of temperature for particular material particles and express the answer in both a material and a spatial description.

8. Obtain the material derivative $\frac{D\theta}{Dt}$ for the motion and temperature field given in the previous exercise.

9. For superposed material and spatial axes, the displacement vector of a body is given by $\mathbf{u} = 4X_1^2\mathbf{e}_1 + X_2X_3^2\mathbf{e}_2 + X_1X_3^2\mathbf{e}_3$. Determine the displaced location of the particle originally at $(1, 0, 2)$.

10. Given the displacement field

$$u_1 = k\left(2X_1 + X_2^2\right); \quad u_2 = k\left(X_1^2 - X_2^2\right); \quad u_3 = 0; \quad k = 10^{-4}$$

   (a) Find the unit elongation and the change of angle for the two material elements $d\mathbf{X}_1 = dX_1\mathbf{e}_1$ and $d\mathbf{X}_2 = dX_2\mathbf{e}_2$ that emanate from a particle designated by $\mathbf{X} = \mathbf{e}_1 - \mathbf{e}_2$.

   (b) Find the deformed position of these two elements $d\mathbf{X}_1$ and $d\mathbf{X}_2$.

11. A unit cube, with edges parallel to the coordinates axes, is given the displacement field:

$$u_1 = kX_1; \quad u_2 = u_3 = 0; \quad k = 10^{-4}$$

Find the increase in length of the diagonal AB (that connects the points: $A\,(0, 0, 0)$ and $B\,(1, 1, 0)$) by using

   (a) the infinitesimal strain tensor.

   (b) geometry.

12. For the velocity field, $\mathbf{v} = kx_2^2\mathbf{e}_1$, find:

   (a) the rate of deformation and spin tensors.

   (b) the rate of extensions of a material element $d\mathbf{x} = (ds)\,\mathbf{n}$ where $n = \left(\frac{\sqrt{2}}{2}\right)(\mathbf{e}_1 + \mathbf{e}_2)$ at $\mathbf{x} = 5\mathbf{e}_1 + 3\mathbf{e}_2$.

13. For the velocity field $\mathbf{v} = \left(\frac{t+k}{1+x_1}\right)\mathbf{e}_1$, find the rates of extension for the following material elements: $d\mathbf{x}_1 = ds_1\mathbf{e}_1; d\mathbf{x}_2 = \left(\frac{ds_2}{\sqrt{2}}\right)(\mathbf{e}_1 + \mathbf{e}_2)$ at the origin at time $t = 1$.

14. Given the following velocity field

$$v_1 = k\left(x_2 - 2\right)^2 x_3; \quad v_2 = -x_1x_2; \quad v_3 = kx_1x_3$$

for an incompressible fluid, determine k such that the equation of mass conservation is satisfied.

15. In a spatial description, the density of an incompressible fluid is given by $\rho = kx_2$. Find the permissible form for the velocity field with $v_3 = 0$, so that the conservation of mass equation is satisfied.

16. Given the velocity field

$$\mathbf{v} = x_1t\mathbf{e}_1 + x_2t\mathbf{e}_2$$

determine how the fluid density varies with time, if in a spatial description it is a function of time only.

17. The state of strain throughout a continuum is specified by

$$\left[\widehat{E}\right] = \begin{bmatrix} X_1^2 & X_2^2 & X_1X_3 \\ X_2^2 & X_3 & X_3^2 \\ X_1X_3 & X_3^2 & 5 \end{bmatrix}$$

Are the compatibility equations for strain satisfied?

18. The strain components are given by

$$E_{11} = \frac{1}{\alpha} f\left(X_2, X_3\right); \quad E_{22} = E_{33} = -\frac{\nu}{\alpha} f\left(X_2, X_3\right); \quad E_{12} = E_{13} = E_{23} = 0$$

Show that for the strains to be compatible $f\left(X_2, X_3\right)$ must be linear.

Solutions:

1a) $x_1 = 1 + 2t^2, x_2 = 2 + t^2, x_3 = 1$; b) $v_1 = 8, v_2 = 4, v_3 = 0, a_1 = 4, a_2 = 2, a_3 = 0$; 2) $v_1 = \frac{16}{15}, v_2 = -\frac{4}{15}, v_3 = 0, a_1 = \frac{8}{15}, a_2 = -\frac{2}{15}, a_3 = 0$; 3) $X_1 = x_1 e^{-t} + x_3\left(e^{-t} - 1\right), X_2 = x_2 + x_3\left(e^{-t} - e^{t}\right), X_3 = x_3$; 4) $v_1 = \left(X_1 + X_3\right) e^{t}, v_2 = X_3\left(e^{t} + e^{-t}\right), v_3 = 0, a_1 = \left(X_1 + X_3\right) e^{t}, a_2 = X_3\left(e^{t} - e^{-t}\right), a_3 = 0$, $v_1 = x_1 + x_3, v_2 = x_3\left(e^{t} + e^{-t}\right), v_3 = 0, a_1 = x_1 + x_3, a_2 = x_3\left(e^{t} - e^{-t}\right), a_3 = 0$; 5a) $\mathbf{v} = (2, 2, 0)$; b) $\mathbf{v} = \left(\frac{2}{5}, \frac{2}{5}, 0\right)$; 6) $\frac{D\theta}{Dt} = -2x_1 e^{-t} - 3x_2 e^{-3t} - 3x_3 e^{-t}$; 7a) $\theta = X_1 + (1 + kt) X_2$; b) $\mathbf{v} = \left(kX_2, 0, 0\right), \mathbf{v} = \left(kx_2, 0, 0\right), \left.\frac{\partial\theta}{\partial t}\right|_{\mathbf{X}\text{ fixed}} = kX_2 = kx_2$; 8) $\frac{D\theta}{Dt} = kx_2$; 9) $\mathbf{x} = (5, 0, 6)$; 10a) unit elongation: 2×10^{-4}, change of angle: 0; 11) $\frac{k}{2}\sqrt{2}$; 12a) $\left[\widehat{D}\right] = \begin{bmatrix} 0 & kx_2 & 0 \\ kx_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \left[\widehat{W}\right] = \begin{bmatrix} 0 & kx_2 & 0 \\ -kx_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; b) $3k$; 13) $-(1 + k), -\frac{(1+k)}{2}$; 14) $k = 1$; 15) $v_1 = v_1\left(x_2, x_3\right), v_2 = v_3 = 0$; 16) $\rho = \rho_0 e^{-t^2}$; 17) No