UNIVERSIDADE da MADEIRA

Continuum Mechanics

Series of exercises 5 - Constitutive equations for simple models of continuum

1. **Example 1** Uma camada de água escorrega, devido a acção do seu peso, por um plano inclinado de inclinação $\alpha = 30\,^\circ$ e de altura $h = 0,5\,\text{m}$. Figura 5.1.1. A corrente de água está limitada pelas paredes laterais e a distância entre as paredes (a largura de escoamento) é $a = 10\,\text{cm}$. A taxa volúmica de escoamento é $G = 2\,\text{l/s}$. A pressão fora da água é atmosférica, $p = p_0$, a espessura da camada no topo do plano $\delta_0 = 1\,\text{cm}$. Despreze a viscosidade da água. Encontre os seguintes parâmetros no ponto mais baixo (saída) do plano: a velocidade de água, a espessura da camada, a pressão exercida pela água sobre o plano.

2. If the Lame constants for a material are: $\lambda = 119,2\,\text{GPa}$; $\mu = 79,2\,\text{GPa}$, find Young's modulus, Poisson's ratio, and the bulk modulus.

3. Given Young's modulus $E_Y = 193\,\text{GPa}$ and shear modulus $\mu = 76\,\text{GPa}$, find Poisson's ratio ν, Lame's constant λ and the bulk modulus k.

4. If the components of strain at a point of structural steel are: $E_{11} = 100 \times 10^{-6}$, $E_{22} = -50 \times 10^{-6}$, $E_{33} = 200 \times 10^{-6}$, $E_{12} = -100 \times 10^{-6}$, $E_{23} = 0$, $E_{13} = 0$. find the stress components ($\lambda = 119,2\,\text{GPa}$; $\mu = 79,2\,\text{GPa}$).

5. Consider the constant stress field below

$$\left[\widehat{T}\right] = \begin{bmatrix} 6 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

   Use the elastic constants $\mu = 79,2\,\text{GPa}$, $\nu = 0,30$:

   (a) Find the strain components.

   (b) Suppose that a sphere of 5 cm radius is under the influence of this stress field, what will be the change in volume of the sphere?

6. **Example 5** Given the following velocity field:

$$v_1 = -k\left(x_1 + x_2\right), \quad v_2 = k\left(x_2 - x_1\right), \quad v_3 = 0, \quad k = 1\,\text{s}^{-1}$$

   for a Newtonian liquid with viscosity, $\mu = 10^{-3}\,\text{Pa}\cdot\text{s}$. For a plane whose normal is in the $\mathbf{e}_1$-direction.

   (a) Find the excess of the total normal compressive stress over the pressure p

   (b) Find the magnitude of the shearing stress.

7. Given the following velocity field in $\text{m}\,\text{s}^{-1}$ for a newtonian incompressible fluid with a viscosity $\mu = 0.96\,\text{mPa}$:

$$v_1 = x_1^2 - x_2^2; v_2 = -2x_1x_2; v_3 = 0$$

   At the point $(1, 2, 1)$ m and on the plane whose normal is in the direction of $\mathbf{e}_1$:

   (a) find the excess of the total normal compressive stress over the pressure p,

   (b) find the magnitude of the shearing stress.

8. Given the velocity field of a linearly viscous fluid

$$v_1 = kx_1, \quad v_2 = -kx_2, \quad v_3 = 0$$

   (a) Show that the velocity field is irrotational.
   (b) Find the stress tensor.
   (c) Find the acceleration field.
   (d) Show that the velocity field satisfies the Navier-Stokes equations by finding the pressure distribution directly from the equations. Neglect body forces. Take $p = p_0$ at the origin.

9. Repeat the preceding exercise for the following velocity field:

$$v_1 = k\left(x_1^2 - x_2^2\right), \quad v_2 = -2kx_1x_2, \quad v_3 = 0$$

10. Show that for the velocity field

$$v_1 = v\left(y, z\right), v_2 = v_3 = 0$$

the Navier-Stokes equations, with $\rho\mathbf{B} = \mathbf{0}$, reduces to

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\mu}\frac{dp}{dx}$$

11. **Example 6** Determine the general form of the stress tensor for an unidimensional flow $\mathbf{v} = (v_1, 0, 0)$ of an incompressible fluid.

Solutions:

1) $v_1 = \sqrt{v_0^2 + 2gx_1\sin\alpha} = 3.7\,\frac{\mathrm{m}}{\mathrm{s}}$; $\delta = \frac{G}{av_1} = 5.4\,\mathrm{mm}$; $p|_{x_2=0,\, x_1\sin\alpha=h} = p_0 + \delta\rho g\cos\alpha \approx 1\,\mathrm{bar}$; 2) $E_Y = 207\,\mathrm{GPa}$; $\nu = 0,30$; $k = 172\,\mathrm{GPa}$; 3) $\nu = 0,27$; $\lambda = 89\,\mathrm{GPa}$; $k = 140\,\mathrm{GPa}$; 4) $\left[\widehat{T}\right] = \begin{bmatrix} 45640 & -15840 & 0 \\ -15840 & 21880 & 0 \\ 0 & 0 & 61480 \end{bmatrix} \times 10^3\,\mathrm{Pa}$; 5a) $\left[\widehat{E}\right] = \begin{bmatrix} 3,34 & 1,26 & 0 \\ 1,26 & -2,33 & 0 \\ 0 & 0 & -0,44 \end{bmatrix} \times 10^{-5}$; b) $\Delta V = 3\times 10^{-3}\,\mathrm{cm}^3$; 6.a) $-T_{11} - p = 2\mu k = 2\cdot 10^{-3}\,\mathrm{Pa}$; b) $T_{21} = -2\mu k = -2\cdot 10^{-3}\,\mathrm{Pa}$; $T_{31} = 2\mu D_{31} = 0$; 7a) $-4\mu = -3,84\times 10^{-3}\,\mathrm{Pa}$; b) $T_{21} = -8\mu\,\mathrm{Pa} = -7,68\times 10^{-3}\,\mathrm{Pa}$; $T_{31} = 0\,\mathrm{Pa}$; 8b) $\left[\widehat{T}\right] = \begin{bmatrix} -p + 2\mu k & 0 & 0 \\ 0 & -p - 2\mu k & 0 \\ 0 & 0 & -p \end{bmatrix}$; c) $\mathbf{a} = k^2\left(x_1, x_2, 0\right)$; d) $p = p_0 - \frac{\rho k^2}{2}\left(x_1^2 + x_2^2\right)$; 9b) $\left[\widehat{T}\right] = \begin{bmatrix} -p + 4\mu k x_1 & -4\mu k x_2 & 0 \\ -4\mu k x_2 & -p - 4\mu k x_1 & 0 \\ 0 & 0 & -p \end{bmatrix}$; c) $\mathbf{a} = 2k^2\left(x_1^3 + 2x_1x_2^2, x_1^2x_2 + x_2^3, 0\right)$; d) $p = p_0 - 2\rho k^2\left(\frac{x_1^4}{4} + \frac{x_2^4}{4}\right)$; 11) $\left[\widehat{T}\right] = \begin{bmatrix} -p & \mu\frac{\partial v_1}{\partial x_2} & \mu\frac{\partial v_1}{\partial x_3} \\ \mu\frac{\partial v_1}{\partial x_2} & -p & 0 \\ \mu\frac{\partial v_1}{\partial x_3} & 0 & -p \end{bmatrix}$.