UNIVERSIDADE da MADEIRA

Mecânica dos Meios Contínuos

Série de exercícios 6 - Sólido Elástico Linear

Exercises marked with $\boxed{\text{www}}$ have their solution in the discipline website.

1. Consider the displacement field for a material half-space that lies to the right of the plane $X_2 = 0$:

$$u_1 = u_3 = 0; \quad u_2 = \phi \sin\left[\frac{2\pi}{l}(X_2 - ct)\right] + \beta \cos\left[\frac{2\pi}{l}(X_2 - ct)\right]$$

   (a) Characterize the movement of the particles in the medium.
   (b) Check if this is an equivoluminal motion.
   (c) Determine ϕ, β, l if the applied displacement on the plane $X_2 = 0$ is given by $\mathbf{u} = a\cos(\omega t)\,\mathbf{e}_2$.
   (d) Determine ϕ, β, l if the applied surface traction on $X_2 = 0$ is given by $\mathbf{t_n} = d\sin(\omega t)\,\mathbf{e}_2$.
   (e) Determine in which conditions the equations of motion are verified (assuming no body forces).

2. Consider the displacement field for a material half-space that lies to the right of the plane $X_2 = 0$:

$$u_1 = u_2 = 0; \quad u_3 = \phi \sin[\beta(X_2 - ct)] + \alpha \cos[\beta(X_2 - ct)]$$

   (a) Characterize the movement of the particles in the medium.
   (b) Check if this is an equivoluminal motion.
   (c) ✠ Determine ϕ, α, β if the applied displacement on the plane $X_2 = 0$ is given by $\mathbf{u} = b\sin(\omega t)\,\mathbf{e}_3$.
   (d) ✠ Determine ϕ, α, β if the applied surface traction on $X_2 = 0$ is given by $\mathbf{t_n} = d\sin(\omega t)\,\mathbf{e}_3$.

3. **Example 3 T** Consider the displacement field

$$u_1 = u_3 = 0; \quad u_2 = \alpha \sin\frac{2\pi}{l}(X_1 - c_T t) + \beta \cos\frac{2\pi}{l}(X_1 - c_T t)$$

   for a material half-space that lies in $X_1 \geq 0$.

   (a) Determine α, β, l if the applied displacement on the plane $X_1 = 0$ is given by $\mathbf{u} = b\sin(\omega t)\;\mathbf{e}_2$.
   (b) Determine α, β, l if the applied surface traction on $X_1 = 0$ is given by $\mathbf{t_n} = d\sin(\omega t)\;\mathbf{e}_2$.

4. **Example 4 T** Consider the displacement field

$$u_1 = u_2 = 0; \quad u_3 = \alpha \cos pX_2 \cos\frac{2\pi}{l}(X_1 - ct).$$

   (a) Show that this is an equivoluminal motion.
   (b) From the equation of motion, determine the phase velocity c in terms of p, l, ρ_0, μ (assuming no body forces).
   (c) This displacement field is used to describe a type of wave in the region $|X_2| \leq h$. Find the phase velocity c if the planes $X_2 = \pm h$ are traction free.

5. $\boxed{\text{www}}$ Consider a linear elastic medium. Assume the following form for the displacement field

$$u_2 = u_3 = 0; \quad u_1 = \varepsilon\{\sin[\beta(X_3 - ct)] + \alpha \sin[\beta(X_3 + ct)]\}$$

(a) Characterize the movement of the particles in the medium.

(b) Determine in which conditions the equations of motion are verified (assuming no body forces).

(c) Suppose that there is a boundary at $X_3 = 0$ that is traction-free. Under what conditions will the above motion satisfy this boundary condition for all time?

(d) Suppose that there is a boundary at $X_3 = l$ that is also traction-free. What further conditions will be imposed on the above motion to satisfy this boundary condition for all time?

6. Same questions as the preceding exercise for the following displacement field

$$u_1 = u_2 = 0; \quad u_3 = \sin\left[\beta\left(X_3 - ct\right)\right] + \alpha \sin\left[\beta\left(X_3 + ct\right)\right]$$

7. A steel ($E_Y = 207\,\mathrm{GPa}$, $\nu = 0,3$) circular bar, $0,61\,\mathrm{m}$ long, $2,54\,\mathrm{cm}$ radius, is pulled by equal and opposite axial forces $P = 44,5\,\mathrm{kN}$ at its ends. Find:

   (a) The maximum normal and shear stresses.

   (b) The total elongation and diameter contraction.

8. A cast iron ($E_Y = 103\,\mathrm{GPa}$, $\nu = 0,25$) bar, $122\,\mathrm{cm}$ long and $3,81\,\mathrm{cm}$ in diameter is pulled by equal and opposite axial forces $P = 89\,\mathrm{kN}$ at its ends. Find:

   (a) The maximum normal and shear stresses.

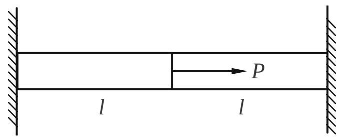
   (b) The total elongation and diameter contraction.

9. A steel ($E_Y = 207\,\mathrm{GPa}$) bar of $3,05\,\mathrm{m}$ length is to be designed to carry a tensile load of $444,8\,\mathrm{kN}$. What should the minimum cross-sectional area be

   (a) if the maximum shearing stress should not exceed $103\,\mathrm{MPa}$ and the maximum normal stress should not exceed $138\,\mathrm{MPa}$?

   (b) if it is further required that the elongation should not exceed $0,127\,\mathrm{cm}$?

10. **Example 7 T** A composite bar, formed by welding two slender bars of equal length and equal cross-sectional area, is loaded by an axial force P. If Young's moduli of the two portions are $E_y^{(1)}$ and $E_y^{(2)}$, find how the applied force is distributed between the two halves.



11. **Example 8 T** Consider a cylindrical bar, with radius $a = 2\,\mathrm{mm}$, and length $l = 1\,\mathrm{m}$. One end of the bar is stuck, the other is twisted through a spanner with length $R = 50\,\mathrm{cm}$. The force applied to the spanner is $F = 10\,\mathrm{kgf}$. The bar is made of steel, $E_y = 2 \times 10^{11}\,\mathrm{Pa}$, $\nu = 0.3$. Determine the angle of rotation of the spanner and the length of the path taken by its end.

Solutions:

1a) O movimento das partículas do meio resulta da propagação de uma onda plana longitudinal, direcção de propagação $\mathbf{e}_2$; b) $E_{kk} \neq 0$ trata-se de um movimento de volume variável; c) $\phi = 0; \beta = a; l = \frac{2\pi}{\omega}c$; d) $\phi = 0; \beta = -\frac{cd}{\omega(\lambda+2\mu)}; l = \frac{2\pi}{\omega}c$; e) $c = \sqrt{\frac{\lambda+2\mu}{\rho_0}}$; 2a) O movimento das partículas do meio resulta da propagação de uma onda plana transversal, direcção de propagação $\mathbf{e}_2$; b) Trata-se de uma onda plana de volume constante pois $E_{kk}=0$; c) $\phi=-b; \alpha=0; \beta=\frac{\omega}{c}$; d) $\phi=0; \alpha=-\frac{dc}{\mu\omega}; \beta=\frac{\omega}{c}$; 3a) $\alpha=-b,\ \ \beta=0,\ \ l=\frac{2\pi c_T}{\omega}$; 3.b) $\alpha=0$; $\beta=-\frac{dl}{2\pi\mu}=-\frac{c_T d}{\mu\omega}$; $l=\frac{2\pi c_T}{\omega}$; 4b) $c=\sqrt{\frac{\mu}{\rho_0}}\sqrt{1+\left(\frac{lp}{2\pi}\right)^2}$; 4c) $p=\frac{\pi n}{h}$; $n=0,1,2,...$; 5a) O movimento das partículas do meio resulta da propagação de duas ondas planas transversais, direcção de propagação $\mathbf{e}_3$; b) $c=\sqrt{\frac{\mu}{\rho_0}}$; c) $\alpha=-1$; d) $\beta=\frac{k\pi}{l}$ com $k=1,2,3,\ldots$; 6a) O movimento das partículas do meio resulta da propagação de duas ondas planas longitudinais, direcção de propagação $\mathbf{e}_3$; b) $c=\sqrt{\frac{(\lambda+2\mu)}{\rho_0}}$; c) $\alpha=-1$; d) $\beta=\frac{k\pi}{l}$ com $k=1,2,3,\ldots$.; 7a) $(T_n)_{\max}=\frac{P}{A}=21,9\,\mathrm{MPa}$; $(T_s)_{\max}=\frac{P}{2A}=11,0\,\mathrm{MPa}$; 7b) $\Delta l\approx 65\,\mu\mathrm{m}$; $\Delta d\approx-1,62\,\mu\mathrm{m}$; 8a) $(T_n)_{\max}=78\,\mathrm{MPa}$; $(T_s)_{\max}=39\,\mathrm{MPa}$; 8b) $\Delta l\approx 9,24\times10^{-4}\,\mathrm{m}$; $\Delta d\approx-7,21\times10^{-6}\,\mathrm{m}$; 9a) $A>3,2\times10^{-3}\,\mathrm{m}^2$; 9b) $A>5,2\times10^{-3}\,\mathrm{m}^2$; 10) $P_2=-\frac{E_y^{(2)}}{E_y^{(1)}+E_y^{(2)}}P,\quad P_1=\frac{E_y^{(1)}}{E_y^{(1)}+E_y^{(2)}}P$; 11) $\theta\left(l\right)=25.36\,\mathrm{rad}$.