On the branching of solutions in the theory of the cathode sheath of a glow discharge

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A study is made of the branching points on the one-dimensional solution of the problem of the cathode region of a glow discharge, i.e., points at which multidimensional solutions branch off from the one-dimensional solution or join up with it. It is shown by numerical calculations that such points exist, and the arrangement of these points is determined. The nature of this arrangement suggests that the multidimensional solutions that branch off or join up at these points describe normal-spot regimes.

The theoretical description of the normal-current-density effect in a glow discharge has attracted considerable interest. A possible approach to this problem is to investigate the multidimensional solutions of the steady-state problem of the cathode sheath. Such a study can be carried out for the entire volume only by means of two-dimensional or three-dimensional numerical calculations. However, useful information of a qualitative nature can be obtained more simply — by analyzing the branching points. Such an analysis is the subject of this paper.

Let us consider the cathode region of a steady flow discharge. The discharge vessel is in the form of a right cylinder, not necessarily circular. We introduce a Cartesian coordinate system x, y, z such that the y axis is along the normal from the cathode surface into the plasma, and we denote by G the region in the xz plane that is occupied by the cathode. The system of equations describing the distribution of the ion and electron densities n and n_e and the electric potential φ are written in the form

\[ \nabla \cdot (\mathbf{v}_i n_i) = a \mathbf{E} n_i - \gamma_i n_i \mathbf{v}_i, \]

\[ \nabla \cdot (\mathbf{v}_e n_e) = 0, \]

\[ \Delta \phi = \frac{1}{\varepsilon} \varepsilon \phi_{\text{dim}} - \Delta \phi \]

(1)

where \( \mathbf{v}_i \) and \( \mathbf{v}_e \) are the mobilities of the ions and electrons, \( a = e \mathbf{E} \), \( \gamma = \frac{1}{\varepsilon} \varepsilon \phi_{\text{dim}} \), the second term on the right-hand side of Eq. (1) models the diffusion or recombination losses, and \( \phi_{\text{dim}} \) is the characteristic frequency of these losses. The parameters \( \mathbf{v}_i, \mathbf{v}_e, \) and \( \phi_{\text{dim}} \) are assumed constant and given, and a is a given function of E: \( a = a(E) \).

Since the drift flux densities of ions and electrons at the cathode surface are related through the secondary emission coefficient \( \gamma \) and the potential is equal to some constant (which, without loss of generality, can be taken to be zero), we have the following boundary conditions at \( y = 0 \):

\[ n_i = \gamma n_e, \]

\[ \phi = 0. \]

(2)

(3)

On the lateral surface of the discharge vessel the normal component of the current density is zero:

\[ \mathbf{v}_i \cdot \mathbf{n} = 0. \]

In a uniform positive column the electric field is constant and is directed along the axis of the vessel; we therefore write the boundary condition at \( y = \infty \) in the form

\[ \phi = E_\infty y + U, \]

(5)

where \( E_\infty \), the absolute value of the field in the column, is determined from the condition \( \alpha(\mathbf{E}_\infty) = \gamma \mathbf{E}_\infty / \mathbf{U}_0 \); \( U \) is a specified parameter having the meaning of the voltage drop across the cathode sheath and is defined as the difference between the value obtained by extrapolating the potential distribution in the positive column to the surface of the cathode and the value of the cathode potential.

We note that the problem (1)-(5) thus formulated does not describe the structure of the transition region between the cathode sheath and the positive column, but for the purposes of this paper this fact is not important.

We divide the second equation in (1) by \( 1 + \delta \) and subtract it from the first. Using the resulting relation and the third equation, we express \( n_0 \) and \( n_i \) in terms of \( \phi \). Substituting these expressions into the second equation in (1) and into condition (2), we obtain \( (K = aE = \phi_{\text{dim}} / \mathbf{U}_0) \)

\[ \nabla \left[ \left( \frac{1}{\varepsilon} \varepsilon \phi_{\text{dim}} - \Delta \phi \right) \nabla \phi \right] = 0. \]

\[ \frac{1}{2(1 + \delta)} \frac{\partial \phi}{\partial \mathbf{x}} - \frac{\partial \phi}{\partial \mathbf{y}} = - \frac{\partial \phi}{\partial \mathbf{y}}. \]

(6)

(7)

Thus Problem (1)-(5) is transformed to the problem (6), (7), (3)-(5) for a single unknown function \( \phi \).

This problem has a well-known one-dimensional solution \( \phi = \phi(y) \), which describes the anomalous discharge regime and a regime (unrealized) corresponding to the descending part of the current-voltage (IV) characteristic. The existence of normal-spot regimes compels us to assume that under certain conditions this problem also has multidimensional solutions \( \phi = \phi(x, y, z) \). Since the normal-spot regimes go over continuously (i.e., in a quasi-stationary manner) to the anomalous regime as the current is increased, it must also be assumed that each of these multidimensional solutions joins up with the one-dimensional solution at least at one point. Thus we can expect that the one-dimensional solution will have branching points, i.e., points at which the multidimensional solutions branch off or join up.

Let us turn to the problem of finding these branching points. Let \( U_0 \) be the value of the voltage drop corresponding to one of these points, and \( \phi_0 = \phi_0(y) \) be the corresponding solution. The solution corresponding to nearby values of the voltage
drop U = U_x + U_z, \[|U_z| \ll |U_x|\) is sought in the form \(\phi = \phi(y) + \phi_1(x, y, z), |\phi_1| \ll |\phi|.\) Substituting these expressions into problem (6), (7), (3)-(3) and keeping terms first of zero order and then of first order, we obtain the following problems for the future \(E_0 = \phi_1, \) and the function \(\phi_1,\) respectively:

\[
E_0 \left[ \frac{1}{K_0} (E_0 E_0')' - \frac{4\kappa J}{\mu_0} \right] - \frac{4\kappa J}{\mu_0} I_y = 0, \quad (\text{8})
\]

\[
y = 0 : E_0 E_0' = \frac{1 + J_y}{1 + J_y} \frac{4\kappa J}{\mu_0}, \quad \phi_1 = 0, \quad y \rightarrow \infty : E_0 \rightarrow E_{\infty},
\]

\[
\frac{\partial}{\partial y} \left[ \frac{1}{K_0} (E_0 E_0') - \frac{4\kappa J}{\mu_0} \right] E_0 E_0' - \phi_1 = 0, \quad (\text{9})
\]

and on the lateral surface \(\partial \phi_1 / \partial n = 0,\) where \(K_0 = K(E_0), a = (d \ln K_0 / \ln E_0), \) \(J_y = \) the value of the current density corresponding to the branching point under consideration, and a prime denotes a total derivative with respect to \(y.\)

Since the point \(U = U_x,\) is a branching point, the linear inhomogeneous problem (9) has a non-unique solution. Consequently, the corresponding homogeneous problem has a nontrivial solution. We seek this solution in the form

\[
\phi_1 = f(y) \Phi(x, z).
\]

Substituting this expression into (9) and performing a separation of variables, we obtain the eigenvalue problems satisfied by the functions \(\Phi\) and \(f:\)

\[
\frac{d^2 \Phi}{dx^2} + \frac{d^2 \Phi}{dz^2} + k^2 \Phi = 0, \quad (\text{10})
\]

with \(\partial \Phi / \partial n = 0\) at the boundary of region \(G,\) and

\[
\text{and}
\]

\[
\text{where} \quad k^2 = \text{the separation constant.}
\]

Let us find the asymptotic behavior of the solutions of equation (11) for \(y \rightarrow \infty.\) Making use of the fact that \(K = (dK/dE_0)_{E=E_0} (E - E_0)\) for \(E = E_0\) and linearizing Eq. (8), we find that for \(y \rightarrow \infty:\)

\[
E_0 \approx E_{\infty} + C e^{-x}, \quad w = \left[ \frac{4\kappa J}{\mu_0} \frac{dK}{dE_0} (E_0) \right]^{1/2}.
\]

Here and below, \(C_1, C_2, ..., C_6\) are arbitrary constants. For \(y \rightarrow \infty,\) Eq. (11) becomes, to a first approximation,

\[
(e^{-(k^2 + w_0)}f)' = 0.
\]

The solution of this equation gives the desired asymptotic behavior:

\[
f = C e^{-x} + C_1 e^{-(\sqrt{R^2 + w_0^2} + y)} + C_2 e^{-(\sqrt{R^2 + w_0^2} - y)} + C_3.
\]

This asymptotic expression is compatible with the last boundary condition in (11) if \(C_1 = C_2 = 0.\) Thus, this boundary condition straightforwardly yields the values of the \(C\) of the integration constants, and problem (11) is closed.

Interestingly, although the component of the potential that depends on \(x\) and \(z\) decays exponentially in the positive column, the analogous component of the axial component of the current density goes to a constant value

\[
J_y = - \frac{k^2 C_1}{\mu_0} J_0 (x, y).
\]

In other words, in the model under consideration the multidimensionality of the parameter distributions in the cathode sheath gives rise to a non-uniformity of the distribution of the axial current and, consequently, of the densities of charge particles over the cross section of the positive column. This circumstance is due to the approximate nature of our allowance for losses in this model.

The procedure for finding the branching points is as follows. One must first find the spectrum, i.e., the set of eigenvalues \(k, k_1, k_2, \ldots\) of problem (11). We note that this spectrum is completely determined by the form of the region \(G,\) i.e., by the transverse cross section of the discharge vessel. Next, the first eigenvalue \(k = k_0\) is substituted into (11). Clearly, for each given value of \(k_0\) all the coefficients of problem (11) can be determined (the function \(E_0\) is determined from the solution of problem (9)), and one can therefore treat (11) as an eigenvalue problem for finding the parameter \(k_0.\) The values of \(k_0\) thus found determine the position of the branching points which correspond to the first eigenvalue of problem (10). At these points the multidimensional solutions whose dependence on \(x\) and \(z\) in the neighborhood of these points is described by the first eigensolution \(\phi_1(x, z)\) of problem (10) join up with the one-dimensional solution. Next, the eigenvalue \(k = k_0\) is substituted into (11) and the position of the branching points corresponding to this value are determined; at these points the solutions described by the second eigensolution \(\phi_2(x, z)\) of (10) join up with the one-dimensional solution. Next, the values \(k_0, k_1,\) etc., are substituted into (11), etc.

In carrying out this procedure it is necessary to solve problems (10), (8), and (11). The solution of problem (10) for regions of simple shape is known. For example, for a discharge vessel in the form of a circular cylinder \((g = \) a circle) the radius \(R\) of the first three (not counting the zeroth) eigenvalues and the corresponding eigenfunctions are

\[
k_1 = 1.84/R, \quad \Phi_1 = J_1(k_1R) \sin \theta, \quad \Phi_2 = J_1(k_2R) \cos \theta,
\]

\[
k_2 = 3.05/R, \quad \Phi_1 = J_1(k_1R) \sin \theta, \quad \Phi_2 = J_1(k_2R) \cos \theta,
\]

\[
k_3 = 3.38/R, \quad \Phi_1 = J_1(k_1R), \quad \Phi_2 = J_1(k_2R),
\]

where \(r\) is the distance from the axis of the vessel, \(\theta\) is the polar angle, and \(\Phi_1, \Phi_2,\) and \(\Phi_3, \Phi_4\) are Bessel functions.

Problems (8) and (11) must be solved numerically; it is convenient to first transform them as follows: in problem (8) – to the new independent variable \(E_0\) and the unknown function \(f = \mu E_0 (\text{4f}),\) and in problem (11) – to the independent variable \(E_0\) and the unknown functions \(f, \psi = E_0.\)
(F−k f):

\[
\frac{dq}{dE} = \frac{\nu K \omega}{4\varepsilon_0} \left( 1 + \frac{1}{q} \right), \quad E_q = E_{m} : q = 0, \quad E_q = E_{w} : q = -\frac{1 - \gamma}{1 + \gamma}. \quad (12)
\]

\[
\frac{d}{dE} \left( q \frac{df}{dE} \right) + b \frac{df}{dE} - q \frac{df}{dE} \frac{d^2 f}{dE^2} + b q = \frac{4\pi e_0 k^2}{\varepsilon_0} f, \quad E_q = E_{m} : q = 0, \quad E_q = E_{w} : q = \frac{1 - \gamma}{1 + \gamma}, \quad (13)
\]

where

\[ q_1 = \frac{4\pi e_0 k}{\varepsilon_0}, \quad q_2 = \frac{K \omega}{\varepsilon_0} \left( \frac{4\pi e_0 k}{\varepsilon_0} + q_1 \right) - \frac{q^2}{E_0}, \quad b = \frac{E_{w}}{K \omega} - a \left( \frac{4\pi e_0 k}{\varepsilon_0} + q_1 \right) + \frac{4\pi e_0 k}{\varepsilon_0}, \quad E_q = E_{w} \text{ at } \gamma = 0, \quad f = 0.
\]

We note that before changing to the new variables, Eq. (11) was transformed in such a way that the coefficients of the system of equations (13) would be regular at the point \( E_q = E_{m} \).

The technique used for the numerical solution was as follows. A certain value of the field \( E_{w} \) on the cathode surface was specified. The corresponding value of \( J_0 \) was determined by numerical integration (using Simpson’s rule):

\[ J_0 = \frac{\pi_0}{4 \varepsilon_0} \left[ \ln \frac{1 + \gamma}{1 + \gamma} - \frac{1 - \gamma}{1 - \gamma} \right] \frac{E_{w}}{E_{m}} K \mu E_{w} \]

Equation (12) with the first boundary condition was solved by the Runge–Kutta method (we note that, after this problem is solved, one can determine the value of the voltage drop corresponding to the given \( E_{w} \)):

\[ U_v = \frac{\pi_0}{4 \varepsilon_0} \left[ \ln \frac{E_{w} - E_{m}}{E_{m}} \right] \frac{E_{w}}{E_{m}} dE_{w}. \]

For solution of problem (12) (a boundary-value problem for a system of two second-order linear equations) we used the method of vector elimination, based on a fourth-order differencing scheme and constituting a generalization of the method of Ref. 5 to the case of a system of equations.

The value of \( E_{w} \) was varied until the determinant of the system of equations obtained after the forward run for determining the values of the functions \( \psi \) and \( f \) and their derivatives at the point \( E_q = E_{w} \) vanished. Even though all the numerical methods used in this study have fourth-order accuracy, the rapid change of the unknown functions made it necessary to choose a large number of mesh points on the segment \( [E_{m}, E_{w}] \).

The calculations were done for conditions modeling a discharge in nitrogen at a pressure of 10 Torr; the values of the transport and kinetic coefficients were taken from Ref. 2: \( \mu_0 = 2 \times 10^{-6} \text{ cm}^2/\text{V-s}, \quad \omega_0 = 3 \times 10^3 \text{ cm}^2/\text{V-s}, \quad a = A e^{-Bp/E}, \quad A_p = 120 \text{ cm}^{-1}, \quad B_p = 3420 \text{ V/cm}, \quad \gamma = 10^3 \text{ s}^{-1}. \) The secondary emission coefficient \( \gamma \) was taken equal to 0.1 and 0.01.
The results of the calculations came out as follows. For \( k \) higher than a certain value \( k_C (k_C = 7.60 \text{ cm}^{-2} \text{ for } \gamma = 0.1, k_C = 2.25 \text{ cm}^{-2} \text{ for } \gamma = 0.01) \), the one-dimensional IV characteristics had no branching points. For \( k \) in the range \( 0 < k < k_C \), there are two branching points. Both of these points lie on the descending part of the IV characteristic, one of them on the steeply falling part and the other on the level part. As \( k \) decreases, the first of these points moves to larger \( U \), while the second moves toward the minimum of the IV characteristic; with increase in \( k \) the branching points move counter to each other and coalesce at \( k = k_C \). Figures 1 and 2 show, for \( \gamma = 0.1 \) and 0.01, respectively, the one-dimensional IV characteristics (the solid curves), the branching points for different values of \( k \) (the filled circles), and the minima of the one-dimensional IV characteristics (the open circles). Figure 1 shows only one branching point each for \( k = 0.25 \text{ cm}^{-2} \) and \( k = 0.5 \text{ cm}^{-2} \), since on the scale of the figure the second branching point for these values coincides with the minimum of the IV characteristic. For the same reason Fig. 2 shows only one branching point for \( k = 0.25 \text{ cm}^{-2} \).

As we see from the figures, in the model used the voltage drop has a finite value at \( j = 0 \). This value can also be found analytically and is equal to

\[
E_0 \left[ \frac{dE}{dU} (E_0) \right]_{\text{max}} \lesssim \frac{1}{T(1 + 1)}.
\]

We note, incidentally, that for \( j \to 0 \) the thickness of the cathode sheath increases without bound, and the model used becomes inapplicable.

Since there are two branching points on the one-dimensional IV characteristic for each \( k \) in the interval \( 0 < k < k_C \), it is natural to assume that one of these points is a branch-off point and the other a joining point for the same multidimensional solution. The hypothetical trend of the IV characteristics corresponding to two such multidimensional solutions is shown by the dashed curves in Fig. 1. Since the relative situation of the branching points in the given problem is analogous to that of the branching points in the problem of thermal constriction, one can assume that the aforementioned multidimensional solutions, like the multidimensional solutions in the model of Ref. 6, describe regimes with a normal spot.

If these hypotheses are correct, this means, in particular, that the diffusive transport of charged particles does not play the governing role in the mechanism of the normal-current-density effect (see the discussion of this question in Ref. 1). Whereas in the model of thermal constriction the onset of the normal-spot regimes is brought on because of saturation of the plasma conductivity with increasing temperature, in the present model the cause is saturation of the ionization coefficient with increasing field.

Interestingly, situations are possible in which the problem under consideration has only three-dimensional solutions in addition to the one-dimensional solution. For example, let us consider the case when the discharge vessel is a circular cylinder of radius 1 cm and the parameters correspond to the conditions of Figs. 1 and 2. If \( \gamma = 0.1 \), then the solutions branching off and joining up at the branching points corresponding to the first three eigenvalues \( k_1 = 1.84 \text{ cm}^{-1} \), \( k_2 = 3.05 \text{ cm}^{-1} \), \( k_3 = 3.84 \text{ cm}^{-1} \) are respectively the three-dimensional solution, the three-dimensional solution periodic in \( \pi \) with period \( \pi \), and the axisymmetric solution. If \( \gamma = 0.01 \), then there are only two branching points on the one-dimensional IV characteristic, corresponding to the first eigenvalue \( k_1, k_2, \ldots > k_C \); in this situation one would expect that the given problem has a one-dimensional and a three-dimensional solution but does not have an axisymmetric solution.

Translated by Steve Torstein

\[^1\text{ Yu. F. Raizer, Teplofiz. Vys. Temp. 24, 986 (1986).}\]
\[^2\text{ Yu. F. Raizer, Principles of Modern Gas-Discharge Physics [in Russian], Nauka, Moscow (1980).}\]