A kinetic derivation of multifluid equations for multispecies nonequilibrium mixtures of reacting gases

M. S. Benilov

Departamento de Física, Universidade da Madeira, Largo do Município, 9000 Funchal, Portugal

(Received 31 May 1996; accepted 2 December 1996)

Derivation of the multifluid equations by means of the Chapman–Enskog method is considered. The forces of inter-species friction and the rates of energy exchange due to elastic collisions are calculated taking into account the effect of the relative motion of the species and of the non-equality of their temperatures on the frequency of inter-species collisions. © 1997 American Institute of Physics. [S1070-664X(97)02403-8]

I. INTRODUCTION

The term “multifluid hydrodynamics” refers conventionally to a continuum description of multispecies nonequilibrium gas mixtures in the framework of which each species of the mixture is regarded as a separate fluid co-existing with the fluids made up of other species. One of the major differences distinguishing the multifluid approach from conventional hydrodynamics theory (i.e., from the model of one fluid with diffusing species) is that the temperatures of the species in the multifluid approach can be unequal. Correspondingly, the system of equations of multifluid hydrodynamics includes, generally speaking, equations of energy for each species, while the system of equations of conventional hydrodynamics includes the energy equation only for the mixture on the whole. Another difference is that mean velocities of the species are governed in multifluid theory by the momentum equations for the species, while the system of equations of conventional hydrodynamics includes the momentum equations for the mixture on the whole and the transport equations for the species (i.e., equations relating mean velocities of the species to the gradients, in particular, to the gradients of the number densities). Note that the transport equations may be written in a form involving the force of the inter-species friction and the gradient of the species pressure, which are present also in the momentum equations for the species. It is essential, however, that the momentum equations involve the convective terms, lacking in the transport equations.

The well-known example of the multifluid approach is the model in which the electrons and heavy particles are treated as separate fluids with different temperatures. Equations for this model are well established. On the other hand, species of comparable particle masses, e.g., ions and neutral atoms in a partially ionized plasma, frequently are also considered as separate fluids in the framework of the multifluid description (e.g., Refs. 1–4 and references therein). For example, such an approach is employed in studies on a theory of a glow discharge (e.g., Refs. 3,5), of cathodic plasma jets in vacuum arcs (e.g., Ref. 6), of a near-cathode region of an arc discharge,7,8 of ion diodes,9 of plasma shocks,10 and of the plasma edge in fusion devices (e.g., Ref. 11 and references therein).

Multifluid equations are usually formulated on the basis of the intuitive concept of mutually penetrating continuums, or, in other terms, of co-existing fluids (e.g., Refs. 1–4). In the framework of such an approach, equations for each species of the mixture are postulated to be in the form of the hydrodynamics (Navier–Stokes) equations for a simple gas with some additional terms accounting for the momentum exchange with other species (the so-called friction forces) and for the energy exchange. As far as a kinetic derivation of multifluid equations for the case of comparable particle masses is concerned, there are papers in which Grad’s moment approach is applied to transport models for scrape-off layer plasmas with account of impurities (e.g., Ref. 12 and references therein). However, a kinetic derivation of multifluid equations based on the method of Chapman–Enskog, which is a conventional tool for the derivation of hydrodynamics equations in the kinetic theory of gases (e.g., Refs. 13–16), as well as in the kinetic theory of plasmas (e.g., Refs. 13, 16), is still absent. Such a derivation is highly desirable; in particular, it will contribute to a better understanding of the physical essence of the multifluid equations and of approximations restricting their applicability.

The friction force for each pair of species is usually assumed to be proportional to their relative mean velocity (e.g., Refs. 2,10,17–19 and references therein). This implies that the frequency of inter-species collisions is not affected by the relative motion of the species, which is true if the velocity of this motion is much smaller than the thermal velocity or, in other words, if the Mach number of this motion is small. However, in conditions in which the multifluid approach is essential the Mach numbers of the relative motion of the species are finite and the temperatures of the species are unequal. (If the Mach numbers of the relative motion and the temperature differences are small, the multifluid approach may be replaced by the conventional hydrodynamics theory.) Hence, the friction force should be calculated taking into account the finite values of the Mach numbers of the relative motion of the species and of the non-equality of their temperatures. This also applies to the terms describing the energy exchange between species. At present, such a calculation has been carried out for the friction force and the energy exchange between the charged-particle species, when the particle interaction is described by the cut-off Coulomb potential.20 It is desirable to derive general results, appli-
cable, in particular, to the friction force and the energy exchange between ions and neutral atoms.

These questions are addressed in this paper. The modified Chapman–Enskog procedure is outlined in Sec. II. The friction forces and the rates of energy exchange between species due to elastic collisions taking into account the finite values of the Mach numbers of the relative motion of the species and of the non-equality of their temperatures are calculated in Sec. III. In the Appendix, a transition is considered from the multifluid equations to the equations of the conventional hydrodynamics (one-fluid) theory.

II. DERIVATION OF MULTIFLUID EQUATIONS BY THE METHOD OF CHAPMAN–ENSKOG

A. System of conservation equations

The Boltzmann equation governing the distribution function $f_a = f_a(r, c, t)$ of a species $\alpha$ in a multicomponent gas mixture is written in the conventional form (e.g., Refs. 13–16)

$$\frac{\partial f_a}{\partial t} + c_a \cdot \nabla f_a + F_a \cdot \nabla f_a = \sum_{\beta} J_{ab} + \Gamma_{a},$$

where $c_a$ is the particle velocity, $F_a$ is a force per unit mass acting on a particle of species $\alpha$ (it is assumed that this force does not depend on $c_a$), the term $J_{ab}$ accounts for elastic collisions with a species $\beta$, the term $\Gamma_{a}$ accounts for production and loss of particles of the species $\alpha$ as a result of reactions (inelastic collisions, with chemical transformations, and radiative processes), and the sum is taken over all species of the mixture.

Equations of conservation of number, momentum, and energy of each species may be obtained by means of a conventional procedure (e.g., Refs. 13–16) of the integration of the Boltzmann equation with the appropriate weights ($m_{a}$, $m_{a} c_{a}$, and $m_{a} c_{a}^{2}/2$, respectively, $m_{a}$ being the mass of a particle of the considered species) and read as

$$\frac{\partial \rho_{a}}{\partial t} + \nabla \cdot (\rho_{a} v_{a}) = m_{a} w_{a},$$

$$\frac{\partial (\rho_{a} v_{a})}{\partial t} + \nabla \cdot (\rho_{a} v_{a} v_{a}) = -\nabla p_{a} + \nabla \cdot \tau_{a} + \rho_{a} F_{a}$$

$$- \sum_{\beta} r_{ab}^{(m)} + w_{a}^{(m)},$$

$$\frac{\partial}{\partial t} \left[ \rho_{a} \left( u_{a} + \frac{v_{a}^{2}}{2} \right) \right] + \nabla \cdot \left[ \rho_{a} \left( u_{a} + \frac{v_{a}^{2}}{2} \right) v_{a} \right]$$

$$= \nabla \cdot ( - \rho_{a} v_{a} + \nabla \cdot \tau_{a} + \rho_{a} F_{a} v_{a}$$

$$- \sum_{\beta} r_{ab}^{(e)} + w_{a}^{(e)}).$$

Here $\rho_{a}$, $v_{a}$, $p_{a}$, $\tau_{a}$, $u_{a}$, and $q_{a}$ are the mass density, the mean velocity, the hydrostatic pressure, the viscous stress tensor, the thermal energy per unit mass, and the heat flux density of the species $\alpha$; $r_{ab}^{(m)}$ and $r_{ab}^{(e)}$ are rates of loss per unit volume of, respectively, momentum and energy of the species $\alpha$ due to elastic collisions with a species $\beta$; $w_{a}^{(m)}$ and $w_{a}^{(e)}$ are rates of change of the number density, momentum, and energy of the species $\alpha$ due to reactions.

$$\rho_{a} = n_{a} m_{a}, \quad n_{a} = \int f_{a} d^{3} c_{a}, \quad v_{a} = \frac{1}{n_{a}} \int c_{a} f_{a} d^{3} c_{a},$$

$$p_{a} = \frac{1}{3} \int m_{a} c_{a}^{2} f_{a} d^{3} c_{a}, \quad \tau_{a} = -\hat{\mathbf{P}}_{a} + \rho_{a} \hat{\mathbf{I}},$$

$$\hat{\mathbf{P}}_{a} = \int m_{a} c_{a} c_{a} f_{a} d^{3} c_{a},$$

$$u_{a} = \frac{1}{n_{a}} \int \frac{1}{2} c_{a}^{2} f_{a} d^{3} c_{a}, \quad q_{a} = \int \frac{1}{2} m_{a} c_{a}^{2} f_{a} d^{3} c_{a}.$$
where species are of the same order of magnitude, then Knudsen equation in the first and second approximations reads, re-

spective to the terms in the right-hand side. Therefore, this equation in the first and second approximations reads, re-

spectively, as

\[ W_a^{(m)} = w_a^{(m)} - m_a v_a w_a = \int m_a C_\alpha d^3 c_\alpha, \]  

\[ W_a^{(e)} = v_a^{(e)} - v_a \cdot w_a^{(m)} - m_a \left( u_a - \frac{v_a^2}{2} \right) w_a \]

The left-hand side of Eq. (15) is of order Knudsen numbers with respect to the terms in the right-hand side. Therefore, this equation in the first and second approximations reads, re-

spectively, as

\[ J_{aa}^{(0)} = 0, \quad J_{aa}^{(1)} = (Df_a)^{(0)}, \]

where

\[ J_{aa}^{(0)} = \int \int \left[ f_a^{(0)}(c_a) f_a^{(0)}(c') - f_a^{(0)}(c_a) f_a^{(0)}(c) \right] \]

\[ \times g \sigma_{aa}(g, \chi') d^2 \Omega_g d^3 c, \]

\[ J_{aa}^{(1)} = \int \int \left[ f_a^{(1)}(c_a) f_a^{(1)}(c') \right. \]

\[ + \left. f_a^{(1)}(c_a) f_a^{(0)}(c) \right] \times g \sigma_{aa}(g, \chi') d^2 \Omega_g d^3 c, \]

\[ (Df_a)^{(0)} = \frac{\partial f_a^{(0)}}{\partial t} + c_a \cdot \nabla f_a^{(0)} + F_a \cdot \frac{\partial f_a^{(0)}}{\partial c_a} - \sum_{\beta \neq \alpha} J_{\alpha \beta}^{(0)} \Gamma_{\alpha}^{(0)}, \]

\[ J_{a\beta}^{(0)} = \int \int \left[ f_a^{(0)}(c_a) f_\beta^{(0)}(c_\beta) - f_a^{(0)}(c_a) f_\beta^{(0)}(c_\beta) \right] \]

\[ \times g \sigma_{a\beta}(g, \chi') d^2 \Omega_g d^3 c_\beta. \]

B. The multifluid model

The mixture can be characterized by the set of Knudsen numbers \( Kn_{a\beta} = \lambda_{a\beta} / L \), where \( \lambda_{a\beta} \) is a characteristic free path for collisions of particles of species \( \alpha \) with particles of species \( \beta \), and \( L \) is a characteristic macroscopic length scale. Assume for brevity that the number densities of all species are of the same order of magnitude, then Knudsen numbers \( Kn_{a\beta} \) and \( Kn_{\beta a} \) are comparable. If \( Kn_{a\beta} \leq 1 \), one can say that species \( \alpha \) and \( \beta \) are coupled. The conventional hydrodynamics theory (model of one fluid with diffusing species) deals with a situation when all species of the mixture are coupled, i.e., when any two species of the mixture are coupled either directly or through a chain of sequentially coupled species. The distribution functions of all species in such a situation are close to a Maxwellian function with a common mean velocity and a common temperature.

In order to derive the multifluid equations, one has to consider a situation when collisions between particles of the same species are frequent, however coupling between different species is absent or incomplete. Various cases are possible: it may be that no two species of the mixture are coupled, or some species may be coupled between themselves but not coupled with other species, etc. For brevity, consideration is restricted in this paper to the case when no two species are coupled. In terms of the Knudsen numbers, the considered case is described by relationships \( Kn_{a\beta} = O(1) \) for \( \alpha \neq \beta \), \( Kn_{a\alpha} \leq 1 \). It should be emphasized that these relationships imply that the cross sections of collisions between particles of the same species are much larger than those of inter-species collisions. It is our aim in this subsection to modify for such a situation the conventional Chapman–Enskog procedure (e.g., Refs. 13–16). The Boltzmann equation for each species, Eq. (1), is written in the form

\[ Df_a = J_{aa}, \]

\[ \frac{\partial f_a}{\partial t} + c_a \cdot \nabla f_a + F_a \cdot \frac{\partial f_a}{\partial c_a} - \sum_{\beta \neq \alpha} J_{\alpha \beta} - \Gamma_{\alpha}. \]

We seek a solution of Eq. (15) as a power series in small parameter \( Kn_{aa} \):

\[ f_a = f_a^{(0)} + Kn_{aa} f_a^{(1)} + \cdots. \]

Here the conventional expression has been used for the integral of elastic collisions (e.g., Refs. 13–16). All designations are also conventional, in particular, \( g \) is the relative velocity of the particles before the collision, \( \sigma_{a\beta}(g, \chi') \) is the differential cross section, \( \chi' \) is the scattering angle, \( \Omega_{g} \) is the solid angle defining a direction of the relative velocity of the particles after the collision. The term \( \Gamma_{\alpha}^{(0)} \) should be evaluated in terms of functions \( f_a^{(0)} \).

The physical sense of the first equation in Eq. (18) is quite clear: a distribution function of each species is governed primarily by collisions of particles of this species between themselves. In the first approximation this function is close to Maxwellian with its own mean velocity \( v_a \) and temperature \( T_a \),

\[ f_a^{(0)}(c_a) = n_a \left( \frac{m_a}{2 \pi k T_a} \right)^{3/2} \exp \left[ - \frac{m_a (c_a - v_a)^2}{2 k T_a} \right], \]

where \( k \) is Boltzmann’s constant.

The second equation in Eq. (18) coincides with the respective equation for a simple gas, provided that the effect of reactions and of elastic collisions with particles of other species on transport of momentum and energy of the species \( \alpha \) is neglected. A solution to this equation is well known (e.g., Ref. 16).

In the framework of the above-described approach, equations of continuity, momentum, and energy of species (2)–(4) should be treated as equations for a simple gas, the presence of other species being taken into account in terms of external sources or sinks of mass, momentum, and energy (terms \( w_a \cdot r_\alpha^m \) and \( w_a \cdot r_\alpha^e \) respectively). These terms may be determined using the Maxwellian distribution functions (23); the terms \( r_\alpha^m \) and \( r_\alpha^e \) accounting for elastic collisions, are calculated in the next section. The hydrostatic pressure and the thermal energy per unit mass also can be determined by means of the Maxwellian functions (23); the
III. SOURCE/SINK TERMS ACCOUNTING FOR ELASTIC COLLISIONS

Substitute Eq. (22) into the first equation in Eq. (8). Employing conventional transformations of the kinetic theory of gases (e.g., Ref. 16, p. 63), one can find

$$\begin{align*}
\left[ r_{ab}^{(m)} \right] &= \int \int \left\{ m_{\alpha}(c_{\alpha} - c_{\beta}) \right\} f_{\alpha}^{(0)}(c_{\alpha})f_{\beta}^{(0)} \times (c_{\beta})g \sigma_{ab}(g, \chi, \Gamma)d^{2}\Omega_{\chi}d^{3}c_{\alpha}d^{3}c_{\beta}, \\
\left[ r_{ab}^{(e)} \right] &= -m_{ab}\left[ \int \int \left\{ \frac{1}{G} \right\} ggQ_{ab}^{(1)}(g)f_{\alpha}^{(0)}(c_{\alpha})f_{\beta}^{(0)} \times (c_{\beta})g d^{2}c_{\alpha}d^{3}c_{\beta} \right],
\end{align*}$$

(24)

where $m_{ab} = m_{\alpha} m_{\beta}/(m_{\alpha} + m_{\beta})$ is the reduced mass. Introduce designations,

$$\begin{align*}
\overline{G} &= \frac{m_{\alpha}T_{\beta}c_{\alpha} + m_{\beta}T_{\alpha}c_{\beta}}{m_{\alpha}T_{\beta} + m_{\beta}T_{\alpha}}, \quad T_{\alpha\beta} = \frac{m_{\beta}T_{\alpha} - m_{\alpha}T_{\beta}}{m_{\alpha} + m_{\beta}}.
\end{align*}$$

(25)

Transforming the integral on the right-hand side of Eq. (25) to variables $G, g$, one finds

$$\begin{align*}
\left[ r_{ab}^{(m)} \right] &= -\frac{m_{ab}n_{\alpha}n_{\beta}}{(2\pi k)^{3}} \frac{m_{\alpha}m_{\beta}}{T_{\alpha\beta}} \frac{3/2}{m_{\alpha}T_{\beta} + m_{\beta}T_{\alpha}} \exp \left\{ -\frac{m_{\alpha}T_{\beta}v_{a}^{2} + m_{\beta}T_{\alpha}v_{b}^{2}}{2kT_{\alpha\beta}} \right\} \\
&\times \left\{ \frac{I_{1}I_{2}}{I_{1} \cdot I_{2} - \frac{m_{\alpha}T_{\beta}(T_{\alpha} - T_{\beta})}{m_{\alpha}T_{\beta} + m_{\beta}T_{\alpha}} I_{1}I_{4} \right\},
\end{align*}$$

(26)

where

$$\begin{align*}
I_{1} &= \int \frac{1}{G} \exp \left\{ -\frac{m_{\alpha}T_{\beta} + m_{\beta}T_{\alpha}}{2kT_{\alpha\beta}} \overline{G}^{2} \right\} m_{\alpha}T_{\beta}v_{a} + m_{\beta}T_{\alpha}v_{b} \sqrt{kT_{\alpha\beta}} \overline{G} d^{2}\overline{G}, \\
I_{2} &= \int \left\{ g \right\} gQ_{ab}^{(1)}(g) \exp \left\{ -\frac{m_{ab}}{2kT_{\alpha\beta}} g^{2} \right\} \times \frac{m_{\alpha}T_{\beta}(T_{\alpha} - T_{\beta})}{m_{\alpha}T_{\beta} + m_{\beta}T_{\alpha}} I_{1}I_{4} \right\},
\end{align*}$$

(27)

The integrals (28) may be evaluated directly. Writing the integrals (29) in spherical polar coordinates with the polar axis in the direction of $(v_{\beta} - v_{a})$ and integrating over the angles, one can express these integrals in terms of quantities $Q_{ab}^{(m)}$ and $Q_{ab}^{(e)}$ defined as

$$\begin{align*}
Q_{ab}^{(m)} &= e^{-M_{ab}^{2}} \int_{0}^{\infty} \chi^{2}e^{-\chi^{2}} Q_{ab}^{(1)} \sqrt{\frac{2kT_{ab}}{m_{ab}}} d\chi, \\
Q_{ab}^{(e)} &= \left\{ \int \left\{ F^{(m)}(2M_{ab}^{2} \chi) \right\} F^{(n)}(2M_{ab}^{2} \chi) d\chi \right\},
\end{align*}$$

(30)

where

$$\begin{align*}
F^{(m)}(y) &= 3\frac{\cosh y - \sinh y}{y^{3}}, \\
F^{(n)}(y) &= \frac{\sinh y}{y}.
\end{align*}$$

(31)

Here $M_{ab}$ is the Mach number of the relative motion of species $\alpha$ and $\beta$. Note that in the case $M_{ab} = 0$, when $F^{(m)} = F^{(e)} = 1$, quantities $Q_{ab}^{(m)}$ and $Q_{ab}^{(e)}$ coincide with the evaluated at the temperature $T_{\alpha\beta}$ average cross section for momentum transfer, $Q_{ab}^{(m)}$, which is frequently used in a kinetic theory of gases (e.g., Ref. 22) instead of the respective $\Omega$-integral.

The rate of momentum exchange (friction force) due to elastic collisions may be finally written as

$$\begin{align*}
r_{ab}^{(m)} &= \frac{n_{\alpha}n_{\beta}kT_{ab}}{nD_{ab}}(v_{a} - v_{\beta}).
\end{align*}$$

(32)

Here $n = \Sigma n_{\alpha}$ is the total number density of the mixture and $D_{ab} = \frac{3\pi}{16} \frac{2kT_{\alpha\beta}}{\pi m_{ab}} \frac{1}{nQ_{ab}^{(m)}}$.

It should be emphasized that the dependence of the friction force on $|v_{a} - v_{\beta}|$ is in a general case non-linear, due to the dependence of $D_{ab}$ on $M_{ab}$.

The rate of energy exchange due to elastic collisions is

$$\begin{align*}
r_{ab}^{(e)} &= \frac{n_{\alpha}n_{\beta}k}{nD_{ab}(m_{\alpha} + m_{\beta})} \\
&\times \left\{ 3s_{ab}kT_{ab}(T_{\alpha} - T_{\beta}) + (m_{\alpha}T_{\beta}v_{a} + m_{\beta}T_{\alpha}v_{b}) \right\} \cdot (v_{a} - v_{\beta}).
\end{align*}$$

(34)

where $s_{ab} = Q_{ab}^{(e)}/Q_{ab}^{(m)}$. $R_{ab}^{(e)}$ differs from $r_{ab}^{(e)}$ by the work of the friction force due to elastic collisions and is given by the formula

$$\begin{align*}
R_{ab}^{(e)} &= \frac{n_{\alpha}n_{\beta}k^{2}T_{ab}}{nD_{ab}m_{\alpha}m_{\beta}} \left\{ 3s_{ab}m_{\alpha}(T_{\alpha} - T_{\beta}) - 2M_{ab}^{2}m_{\beta}T_{\alpha} \right\}.
\end{align*}$$

(35)

Note that quantities $R_{ab}^{(e)}$ are not antisymmetric with respect to indices $\alpha$ and $\beta$, in contrast to $r_{ab}^{(e)}$ and $r_{ab}^{(m)}$.

Thus, the rate of energy exchange due to elastic collisions between species $\alpha$ and $\beta$ is governed by two kinetic coefficients: $D_{ab}$, which is the same one that governs the momentum exchange, and $s_{ab}$. The rate of energy exchange
contains two terms, apart from the term accounting for the work of the friction force: the term in the temperature difference and the term in the squared mean relative velocity.

In the limit case $M_{a\beta}\ll 1$, coefficients $D_{a\beta}$ to the accuracy of order $M_{a\beta}^2$ do not depend on $M_{a\beta}$ and coincide with the coefficients of binary diffusion of the species $a$ and $\beta$ at the temperature $T_{a\beta}$ in the conventional hydrodynamics, evaluated by means of the kinetic theory of gases in the first approximation in expansion in the Sonine polynomials in the method of Chapman–Enskog [e.g., Eq. (68.3) in Ref. 14 or Eq. (7.3-38a) in Ref. 16]. The friction force is linear in $|v_a - v_\beta|$ and Eq. (32) in the specific case $T_{a\beta}=T_{\beta}$ will coincide with the expression for the friction force.\footnote{17} The second term in the square brackets in (35), being of order $M_{a\beta}^2$ relative to the first term, is unessential. $s_{a\beta}=1$ in the considered limit, to the accuracy also of order $M_{a\beta}^2$. The rate of energy exchange for given $T_{a\beta}$ is proportional to $(T_{a\beta}-T_{\beta})$.

In the opposite limit case, $M_{a\beta}\gg 1$, one finds in the first approximation,

$$
\begin{aligned}
\left\{ \begin{array}{l}
Q_{a\beta}^{(m)} \\
Q_{a\beta}^{(e)}
\end{array} \right\} &= \left( \begin{array}{l}
\frac{3M_{a\beta}}{M_{a\beta}/4} \\
\frac{3M_{a\beta}}{M_{a\beta}/4}
\end{array} \right) Q_{a\beta}^{(1)}(|v_a - v_\beta|),
\end{aligned}
$$

(36)

and (32) and (34) assume the form

$$
\begin{aligned}
\left\{ \begin{array}{l}
\mathbf{v}_{a\beta}^{(m)} \\
\mathbf{v}_{a\beta}^{(e)}
\end{array} \right\} &= m_{a\beta}n_{a\beta}R |v_a - v_\beta| Q_{a\beta}^{(1)}(|v_a - v_\beta|) \\
& \times \left\{ \frac{m_av_a + m_\beta v_\beta}{m_a + m_\beta} \right\} (v_a - v_\beta),
\end{aligned}
$$

(37)

which describes momentum and energy exchange between two monoenergetic particle beams.

Quantities $Q_{a\beta}^{(m)}$ and $Q_{a\beta}^{(e)}$, governing the friction force and the rate of energy exchange due to elastic collisions, depend on $T_{a\beta}$ and $M_{a\beta}$. For an arbitrary cross section for momentum transfer, these quantities may be calculated by means of a numerical integration. For some model cross sections, they may be found also analytically. We consider three examples: the case when particles of the species $a$ and $\beta$ interact as rigid spheres ($Q_{a\beta}^{(1)}=$const), the case when they interact as Maxwell molecules [constant frequency of momentum transfer, $gQ_{a\beta}^{(1)}(g)$=const], and the case of the cut-off Coulomb potential [$Q_{a\beta}^{(1)}(g) = 4\pi(Z_aZ_\beta e^2/4\pi\epsilon_0m_{a\beta}g^3)\ln\Lambda$, where $Z_a$ is the charge number of the species $a$, $\epsilon_0$ is the dielectric permittivity, and $\ln\Lambda$ is the Coulomb logarithm]. Appearing integrals may be conveniently evaluated with the use of Eqs. (20) of the work.\footnote{23} The results are

$$
\begin{aligned}
Q_{a\beta}^{(m)} &= E_{a\beta}^{(1)} \frac{3}{8} \left( \frac{\sqrt{\pi}}{M_{a\beta}} \right) \left( 1 + M_{a\beta}^2 - \frac{1}{4M_{a\beta}} \right) \text{erf}M_{a\beta} \\
& + \left( 1 + \frac{1}{2M_{a\beta}^2} \right) e^{-M_{a\beta}^2},
\end{aligned}
$$

(38)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Non-dimensional kinetic coefficients $\xi$, $\zeta$ governing momentum and energy transfer in elastic collisions. 1–3: $\xi$, 4–6: $\zeta$. 7,8: asymptotic behavior of coefficient $\xi$ at large $M_{a\beta}$. 9: asymptotic behavior of coefficient $\zeta$ at large $M_{a\beta}$. 1,4,7: the model of rigid spheres, 2,5: the model of Maxwell molecules, 3,6,8: the cut-off Coulomb potential.}
\end{figure}

\begin{align}
Q_{a\beta}^{(e)} &= \frac{3}{4M_{a\beta}} \left( \frac{\sqrt{\pi}}{M_{a\beta}} \right) \text{erf}M_{a\beta} + \frac{M_{a\beta}^2}{3} Q_{a\beta}^{(m)},
\end{align}

(39)

for rigid spheres (here $Q_{a\beta}^{(1)}=Q_{a\beta}^{(1)}$),

\begin{align}
Q_{a\beta}^{(m)} &= \frac{3}{2M_{a\beta}} \left( \frac{\sqrt{\pi}}{2} \right) \text{erf}M_{a\beta} - M_{a\beta}^2 \text{erf}M_{a\beta} - M_{a\beta}^2,
\end{align}

(40)

for Maxwell molecules [here $\bar{Q}_{a\beta}^{(1)}$ = $\sqrt{\pi/m_{a\beta}/128kT_{a\beta}Q_{a\beta}^{(1)}(g)}$, and

\begin{align}
Q_{a\beta}^{(e)} &= \frac{3}{2M_{a\beta}} \left( \frac{\sqrt{\pi}}{2} \right) \text{erf}M_{a\beta} - M_{a\beta}^2 \text{erf}M_{a\beta} - M_{a\beta}^2,
\end{align}

(41)

for the cut-off Coulomb potential [here $\bar{Q}_{a\beta}^{(1)} = (1/32\pi) \times (Z_aZ_\beta e^2/\epsilon_0kT_{a\beta})^2\ln\Lambda$]. Note that Eqs. (41) conform to respective results.\footnote{26}

In the specific cases considered, non-dimensional kinetic coefficients $\xi_{a\beta}$ at large $M_{a\beta}$ is governed by the first equation in Eq. (36) and is $\xi_{a\beta} = 3\sqrt{\pi/M_{a\beta}^2}$ for the case of rigid spheres, $\xi_{a\beta} = 1$ for Maxwell molecules, and $\xi_{a\beta} = 3\sqrt{\pi/4M_{a\beta}^3}$ for the cut-off Coulomb potential. Dependencies for the cases of rigid spheres and the cut-off Coulomb potential are depicted in Fig. 1 by the broken lines. The third broken line in Fig. 1 represents the asymptotic behavior of coefficient $\xi_{a\beta}$ at large $M_{a\beta}$, governed by the second equation in Eq. (36).
In the case of rigid spheres, function $\xi_{\alpha\beta}(M_{ab})$ is increasing, which means that the effect of finite values of the Mach number of the relative motion of the species results in an increase of the friction force. In the case of Maxwell molecules, this function is unity. The physical sense of this result is quite clear: the frequency of momentum transfer is constant and is not affected by the relative motion of the species. In the case of the cut-off Coulomb potential, the effect of finite values of the Mach number of the relative motion of the species results in a decrease of the friction force (Ref. 20 and references therein).

IV. CONCLUDING DISCUSSION

It has been shown that in order to derive the multifluid equations by means of the method of Chapman–Enskog the Knudsen numbers characterizing collisions of particles of any species between themselves should be assumed to be small and Knudsen numbers characterizing collisions of particles of different species should be assumed to be finite. The physical sense of this result is quite clear: the multifluid approach is appropriate in situations when collisions of particles of each species between themselves are frequent, however coupling between different species is weak.

Generally speaking, the above-mentioned assumptions concerning the Knudsen numbers imply that cross sections of collisions between particles of different species are much smaller than those of collisions between particles of the same species. For example, in the case of a strongly ionized plasma the multifluid approach with a separate description of ions and neutral atoms is applicable provided that $Q_{\text{ab}}/Q_{\text{ab}} > Q_{\text{ab}}$, where $Q_{\text{ab}}$ and $Q_{\text{ab}}$ are cross sections of collisions between particles of different species and $Q_{\text{ab}}$, respectively. (It is assumed that densities of ions and of neutral atoms are comparable.) The Coulomb cross section is usually much larger than the cross section charged particle–neutral, hence the first condition is usually satisfied. However, the cross section neutral–neutral usually is not as large as compared to the cross section ion–neutral, especially if the latter is governed by charge exchange. Hence, if Knudsen numbers $Kn_{ii}$ and $Kn_{i0}$ satisfy the above-mentioned conditions [i.e., if $Kn_{ii} < 1$ and $Kn_{i0} = O(1)$], then $Kn_{i0} > O(1)$ and the fluid approximation, while being applicable to the ion species, is strictly speaking, not applicable to the neutral species. An example of such a situation can be found in Ref. 8. On the other hand, one is usually interested primarily in parameters related to the ion fluid, such as the ion current (e.g., Ref. 8), and these parameters are affected by a way of the description of the neutral species only indirectly. In such situations the question of description of a neutral species is not crucial and one can, in a first approximation, employ the multifluid model.

The rates of exchange of momentum and energy between species due to reactions, one has to specify a type of reactions occurring in the considered mixture. For example, in Ref. 23 formulas have been obtained that are applicable to inelastic collisions, to binary collisions with chemical transformations in which particles of species $\alpha$ disappear, and to binary collisions with chemical transformations in which particles of species $\alpha$ are produced provided that the reaction products are two particles.

An important question which always arises when multifluid equations are concerned is that of the importance of the viscous stress terms. It can be shown that these terms, in the conditions of applicability of the multifluid theory, are unessential in one-dimensional cases, but may become important in multidimensional cases.

It is natural to expect that a transition exists between the multifluid approach and conventional hydrodynamics (one-fluid) theory: in a certain limit case, the multifluid equations should reduce to those of the hydrodynamics theory. Let us identify this limit case. The multifluid equations have been derived by the method of Chapman–Enskog in Sec. II B under the suppositions that Knudsen numbers characterizing collisions of particles of each species between themselves, $Kn_{aa}$, are small, and Knudsen numbers characterizing inter-species collisions, $Kn_{ab}$, are of order unity. The last supposition may be eased: it is essential only that $Kn_{ab}$ be much larger than $Kn_{aa}$. Thus, the area of applicability of the multifluid approach may be described as $Kn_{aa} \ll Kn_{ab}$. 1.

The conventional hydrodynamics (one-fluid) theory is applicable if all Knudsen numbers are small. Both the multifluid approach and the conventional hydrodynamics theory are valid in the case $Kn_{aa} \ll Kn_{ab} \approx 1$. Hence, the multifluid equations must coincide with the conventional hydrodynamics equations provided that limits corresponding to this case are applied to both sets of equations. As far as the multifluid equations are concerned, the appropriate limit is $Kn_{ab} \rightarrow 0$. The hydrodynamics equations are formulated, generally speaking, for the case when all Knudsen numbers are comparable, so the limit to be applied to these equations is $Kn_{ab} / Kn_{aa} \rightarrow \infty$. The latter is equivalent to setting $Q_{ab}/Q_{aa} \rightarrow 0$, i.e., to assuming that cross sections of interspecies collisions are much smaller than those of collisions between particles of the same species. It is shown in the Appendix that a limit form of the multifluid equations for the case $Kn_{ab} \rightarrow 0$ indeed coincides with a limit form of the conventional hydrodynamics equations for the case $Q_{ab}/Q_{aa} \rightarrow 0$.

It is expected that the present analysis will prove useful, in particular, in the modeling of near-cathode layers in arc discharges, where the multifluid approach is essential. 7 8

ACKNOWLEDGMENTS

The author appreciates the financial support of FEDER and of the program Praxis XXI. Part of the work has been carried out in the Ruhr-Universität Bochum and is supported by the Alexander von-Humboldt Stiftung.
APPENDIX: THE ONE-FLUID LIMIT

We rewrite Eqs. (9) and (10) in terms of the diffusion velocity of species, \( \mathbf{V}_a = \mathbf{v}_a - \mathbf{v} \),

\[
\frac{d(p_a / \rho)}{dt} + \nabla \cdot (p_a \mathbf{V}_a) = m_a \mathbf{w}_a. \tag{A1}
\]

\[
\frac{d\mathbf{V}_a}{dt} + \rho_a (\mathbf{V}_a \cdot \nabla) \mathbf{v} + \rho_a (\mathbf{V}_a \cdot \nabla) \mathbf{V}_a
\]

\[
= -\nabla p_a + \nabla \cdot \mathbf{\tau} + \rho_a \mathbf{F}_a - \sum_{\beta} \mathbf{r}^{(m)}_{a\beta} + \mathbf{W}^{(m)}_a
\]

\[
+ \frac{\rho_a}{p} \left( \nabla p - \nabla \cdot \mathbf{\tau} + \nabla \cdot \sum_{\beta} \mathbf{\tau}_\beta + \sum_{\beta} \rho_\beta \mathbf{V}_\beta \cdot \mathbf{V}_\beta - \sum_{\beta} \rho_\beta \mathbf{F}_\beta \right). \tag{A2}
\]

where \( \rho, \mathbf{v}, \) and \( p \) are the mass density, the mean mass velocity, and the pressure of the mixture, and \( d/dt = d/dt + (\mathbf{v} \cdot \nabla) \) is the convective derivative for the mixture on the whole.

Assume for simplicity that the characteristic time scale is of the order of \( L/v_0 \), then the non-stationary terms are comparable to the convective terms. In order to estimate the order of magnitude of the diffusion velocities, one can equate orders of magnitude of \( \nabla \rho_a \) and \( \mathbf{r}^{(m)}_{ab} \) on the right-hand side of Eq. (A2). It follows that \( \mathbf{V}^0 = O(Kn_{ab} C^0) \). The order of magnitude of the Mach number \( M_{ab} \) of the relative motion of the species may be estimated as \( |\mathbf{V}_a - \mathbf{V}_\beta|/C^0 \). Hence, \( M_{ab} = O(Kn_{ab}) \). The terms of Eq. (A2) are of the order of, respectively, \( Kn_{ab} C^0 \), \( Kn_{ab} C^0 \), \( Kn_{ab} C^0 \), \( Kn_{ab} C^0 \), \( |T_a - T_\beta| / T^0 \), \( Kn_{ab} C^0 \). It follows that \( |T_a - T_\beta| / T^0 \rightarrow O(Kn_{ab}) \). Hence, all plasma species in the limit case \( Kn_{ab} \rightarrow 0 \) in the first approximation have the same temperature, which will be designated by \( T \).

The terms on the left-hand side of Eq. (A2) are of the order of, respectively, \( Kn_{ab} v_0 / C^0 \), \( Kn_{ab} v_0 / C^0 \), \( Kn_{ab} v_0 / C^0 \), \( Kn_{ab} \), \( [T_a - T_\beta] / T_0 \), \( Kn_{ab} C^0 \). Hence, all plasma species in the limit case \( Kn_{ab} \rightarrow 0 \) in the first approximation have the same temperature, which will be designated by \( T \).

The terms of Eq. (A2) are of the order of, respectively, \( Kn_{ab} v_0 / C^0 \), \( Kn_{ab} v_0 / C^0 \), \( Kn_{ab} v_0 / C^0 \), \( Kn_{ab} \), \( [T_a - T_\beta] / T_0 \), \( Kn_{ab} C^0 \). Hence, all plasma species in the limit case \( Kn_{ab} \rightarrow 0 \) in the first approximation have the same temperature, which will be designated by \( T \).

The terms of Eq. (A2) are of the order of, respectively, \( Kn_{ab} v_0 / C^0 \), \( Kn_{ab} v_0 / C^0 \), \( Kn_{ab} v_0 / C^0 \), \( Kn_{ab} \), \( [T_a - T_\beta] / T_0 \), \( Kn_{ab} C^0 \). Hence, all plasma species in the limit case \( Kn_{ab} \rightarrow 0 \) in the first approximation have the same temperature, which will be designated by \( T \).

\[
\nabla p_a - \rho_a \mathbf{F}_a - \frac{\rho_a}{p} \left( \nabla p - \sum_{\beta} \rho_\beta \mathbf{F}_\beta \right)
\]

\[
= \sum_{\beta} \frac{n_a n_\beta T}{D_{ab}} (\mathbf{V}_\beta - \mathbf{V}_a). \tag{A3}
\]

Since \( M_{ab} \rightarrow 0 \) in the considered limit case, quantities \( D_{ab} \) in these equations represent the conventional hydrodynamics coefficients of binary diffusion, calculated by means of a conventional kinetic theory of gases in the first approximation in expansion in the Sonine polynomials in the method of Chapman–Enskog. A comparison shows that these equations coincide with written in the first approximation Stefan–Maxwell equations (e.g., Refs. 15, 24, 25), relating the diffusion velocities and the diffusion forces in a multispecies gas mixture in the conventional hydrodynamics (one-fluid) theory. It should be noted that written in the first approximation Stefan–Maxwell equations, as well as Eqs. (A3), do not contain thermal-diffusion terms.

Equations of conservation of momentum and energy for the mixture on the whole can be obtained by summing up Eqs. (3) and (4) over various species. Applying to the obtained equations the limit \( Kn_{ab} \rightarrow 0 \), one arrives at the momentum and energy equation of the conventional hydrodynamics theory with the viscosity and thermal-conductivity coefficients \( \eta = \sum_a \eta_a \) and \( \kappa = \sum_a \kappa_a \), respectively, where \( \eta_a \) and \( \kappa_a \) are the viscosity and thermal-conductivity coefficients of the species \( a \). In the case when cross sections of collisions between particles of different species are much smaller than those of collisions between particles of the same species, thus introduced \( \eta \) and \( \kappa \) coincide with the hydrodynamics viscosity and thermal-conductivity coefficients of the mixture calculated by means of the conventional kinetic theory of gases in the first approximation in expansion in the Sonine polynomials in the method of Chapman–Enskog. This can be proved by applying the limit \( Q_{ab} / Q_{aa} \rightarrow 0 \) to expressions for the viscosity and thermal-conductivity coefficients of a multicomponent mixture in the kinetic theory of gases (e.g., Eqs. (70.2) and (69.2) of Ref. 14 or Eqs. (7.3-82) and (7.3-87) of the Ref. 16).