Bifurcation analysis of the corona discharge on a negative electrode

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Abstract

A bifurcation analysis of the cylindrically symmetric solution obtained in the framework of the steady-state model of the negative corona discharge in air at atmospheric pressure has been carried out. No bifurcations have been found, which indicates that current contraction probably does not occur in the considered model. It follows that the current-free stripes on the corona cathode surface, observed by previous authors in two- and three-dimensional computer simulation and in experiment, are due to the specific discharge geometry rather than current contraction. © 1997 Elsevier Science B.V.

1. Introduction

Current structures on electrodes due to current contraction in a near-electrode region have been known in gas discharge physics for a long time. The best known example are normal cathode spots in glow discharges and spots on electrodes in arc discharges (see, e.g., Ref. [1] and references therein). Self-consistent theoretical models began to appear only in the last decade: normal cathode spots in the glow discharges have been described numerically [2,3], arc spots have been described numerically [4–6] and asymptotically [6–8].

It is well-known that current spots occur also on electrodes in the corona discharges [9]. Macroscopic current structures of another type were observed on spiral corona cathodes in an electrostatic precipitator configuration in recent experiments [10]: part of the cathode was dark with bright points, another part was luminous with longitudinal dark strips. The bright points seem to correspond to the spots on the surface of a corona cathode in electron-attaching gases [9].

The dark strips were interpreted [10] as current-free zones. Later, the current-free strips on cylindrical and spiral cathodes have been observed in two- and three-dimensional steady-state computer simulation [11].

The basic question concerning these current-free strips is whether they are due to (1) current contraction or (2) the specific discharge geometry. The second hypothesis implies that there is no regime with a more or less uniform current distribution over the cathode surface in the discharge configuration considered, thus the regime with the strips is the only one possible in this situation. The first hypothesis implies that the geometrical factor is not decisive and there is a regime with a uniform current distribution over the cathode surface, however it is unstable and does not appear in the physical and computer experiments, while the regime with the strips is stable and does appear. If this hypothesis is correct, the current-free strips are self-organized structures, just as the normal spots on glow cathodes and the arc spots. Note that the question of which of the above two hypotheses is correct, being
of interest in itself, is also practically important. For example, if the first hypothesis is correct, then the current transfer in the vicinity of the (wire) cathode is governed mainly by the total current in a considered cross section, hence the area of the current-free (i.e., inactive) part of the cathode is affected by the discharge geometry and the applied voltage in the first approximation only through the current per wire.

In the computer simulation \[11\], regimes of both types (with a nearly uniform distribution of the electric current over the cathode surface and with current-free strips) have been observed, depending on the parameters of the discharge. This indicates that the effect of the geometrical factor is weak, which supports the first hypothesis. Note that the equations considered in Ref. \[10\] are not very different from those describing the normal spots on glow cathodes: both sets of equations include the Poisson equation and the equation(s) of continuity of mobility-dominated flux of charge carriers. This also conforms to the first hypothesis.

From the mathematical point of view, the above question is that of multiplicity of solutions. Numerical calculation of multiple solutions is difficult even if all solutions are one-dimensional \[12\], which is not the case in the problem of contracted current transfer in a near-electrode region. Numerical calculation of multiple solutions of this problem has been considered in Ref. \[13\]; a special approach based on the bifurcation theory has been suggested. In particular, it was found in Ref. \[13\] that the bifurcation analysis itself provides valuable information on regimes with current contraction. In Ref. \[14\], the bifurcation analysis of the normal current spot on the cathode of the glow discharge has been performed. The aim of this brief communication is to perform the bifurcation analysis of the corona discharge on a negative electrode and to consider on the base of this analysis whether the current-free strips are due to current contraction or due to the specific discharge geometry.

2. The model

We consider the computational model of a negative corona discharge in air at atmospheric pressure suggested in Refs. \[11,15\]. The model is based on a steady-state description of the discharge and is simple enough to be tractable in multidimensional situations. In brief, this model may be described as follows. It is assumed that ionization of neutral particles is localized in a thin active zone adjacent to the corona cathode surface. The positive ions drift to the surface. The electrons attach rapidly to molecules to form negative ions which leave the active zone and drift to the grounded electrode. The plasma outside the active zone is described by the conventional system of equations \[16\] consisting of the Poisson equation, the equation of current continuity, and Ohm's law,

$$
\varepsilon_0 \Delta \varphi = e n_-, \quad \nabla \cdot j = 0, \quad j = -e \mu_- n_- \nabla \varphi. \quad (1)
$$

Here $\varphi$ designates the electrostatic potential, $n_-$ is the number density of the negative ions, $j$ is the electric current density, $\mu_-$ is the mobility of the negative ions (a given constant), $\varepsilon_0$ is the dielectric permittivity of vacuum, and $e$ is the electron charge.

The boundary conditions for these equations at the edge of the active zone are

$$
S_1 : \quad \varphi = -U, \quad e n_- = c \tilde{a}. \quad (2)
$$

Here $S_1$ is the surface representing the edge of the active zone, $U$ is the voltage applied to the corona cathode, $c$ is a given constant, and $\tilde{a}$ is the effective ionization coefficient ($\tilde{a} = \alpha - \eta$, where $\alpha$ and $\eta$ are the ionization and attachment coefficients, respectively) considered as a given function of the local electric field. The first condition implies that a voltage drop in the active zone is neglected as compared to the total voltage applied. The second condition accounts in an approximate way for the production of charge carriers in the active zone and is central in the model \[11,15\]. Supposing that the thickness of the active zone is much smaller than the dimensions of the corona cathode, one can assume in calculations that the surface $S_1$ coincides with the cathode surface.

Designate by $S_2$ the surface of a (grounded) outer electrode and by $S_3$ an insulating part of the inner surface of the discharge tube. The respective boundary conditions are

$$
S_2 : \quad \varphi = 0, \quad S_3 : \quad j_n = 0. \quad (3)
$$

Here and below $n$ designates the normal.

It is convenient to transform the problem (1)–(3) to a single unknown function $\varphi$:

$$
\nabla \cdot (\Delta \varphi \nabla \varphi) = 0, \quad (4)
$$

2. The model
3. Bifurcation analysis

3.1. The approach

In this section, the bifurcation analysis of the problem (4), (5) will be carried out. The procedure is similar to that applied in Ref. [14] to the problem of the normal current spot on a glow cathode.

We assume that the current-free strips are due to current contraction, i.e., that the first hypothesis described in Section 1 is correct. This means that the geometrical factor is not decisive and the considered problem has a solution describing the regime with a more or less uniform current distribution over the cathode surface, and one or more solutions describing regimes with the current contraction (with current-free strips).

In the simplest case, one can completely exclude the geometric factor by considering a cylindrical or spherical geometry. For definiteness, we shall consider the cylindrical geometry, i.e., assume that the discharge tube consists of two concentric cylindrical electrodes and the insulating ends. Then surfaces \( S_1, S_2, \) and \( S_3 \) are described by equations \( r = a, r = b, \) and \( z = 0, H, \) respectively, where the cylindrical coordinates \( r, \theta, z \) related to the axis of the electrodes are employed, \( r \) being the distance from the axis, \( \theta \) the azimuthal angle, and \( z \) the coordinate measured along the axis; \( a \) is the radius of the corona cathode; \( b \) is the radius of the outer (grounded) anode; \( H \) is the height of the discharge tube. The regime with a uniform current distribution over the cathode surface is described by a one-dimensional solution \( \varphi = \varphi (r) \), and solutions describing regimes with current contraction are multidimensional: \( \varphi = \varphi (r, \theta, z) \). Thus, the hypothesis that the current-free strips are due to current contraction amounts to the claim that the problem (4), (5) in the cylindrical configuration has one- and multidimensional solutions simultaneously.

In accordance to the results of Refs. [13,17], it is natural to assume that regimes with current contraction, if they exist, under variation of the voltage continuously join the regime with a uniform current distribution over the cathode surface. In mathematical terms, this means that the one-dimensional solution has bifurcation points in which multidimensional solutions branch off. Thus, one should find the one-dimensional solution and carry out its bifurcation analysis.

3.2. One-dimensional solution

The one-dimensional solution of the problem (4), (5) in the cylindrical geometry may be readily found analytically (see, e.g., Ref. [18] and references therein) and written in the form

\[
E = E_w \sqrt{\frac{a^2}{r^2} + A \left( 1 - \frac{a^2}{r^2} \right)},
\]

(6)

Here \( E = d \varphi / dr \) is the electric field, \( A = ac \alpha / e_0 E_w \) and \( E_w = E(a) \) is the electric field strength at the cathode surface (or, more accurately, at the edge of the active zone), which is related to the applied voltage \( U \) by the equation

\[
U = E_w a \left( 1 + A (I^2 - 1) - 1 + B \right),
\]

(7)

where \( \Gamma = b/a \) and

\[
B = \sqrt{1 - A \ln \frac{\Gamma (\sqrt{1 - A} + 1)}{\sqrt{1 - A} + \sqrt{1 + A (I^2 - 1)}}}
\]

for \( A \leq 1 \),

\[
B = -A \arctan \frac{\sqrt{1 + A (I^2 - 1) - 1}}{\sqrt{1 + A (I^2 - 1) + A - 1}}
\]

for \( A \geq 1 \).

(8)

Note that the quantity \( \bar{\alpha} \) should be considered as a known function of \( E_w \).

3.3. Bifurcation analysis

Designate by \( U_0 \) and by \( \varphi_0 (r) \), respectively, the value of the applied voltage and the distribution of the electrostatic potential corresponding to one of the bifurcation points. Represent solutions in the vicinity of this point (i.e., for \( U \) close to \( U_0 \)) in the form \( \varphi (r, \theta, z) = \varphi_0 (r) + \psi (r, \theta, z) \). Writing Eq. (4) first for the function \( \varphi = \psi + \varphi_0 \), then for the function \( \varphi_0 \), and subtracting the obtained relationships, one gets

\[
\nabla \cdot \left( \Delta \varphi_0 \nabla \psi + \Delta \psi \nabla \varphi_0 + \Delta \psi \nabla \psi \right) = 0.
\]

(9)
In the vicinity of the bifurcation point function $\psi$ is small and this equation may be linearized (the term quadratic in $\psi$ may be dropped). One gets after transformations

$$
\frac{\partial}{\partial r} \left[ E_0 \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + w \frac{\partial \psi}{\partial r} + rE_0 \left( \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \right] + w \left( \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} \right) = 0, \tag{10}
$$

where $E_0 = d\varphi_0/dr$ and $w = d (rE_0) / dr$.

Boundary conditions for function $\psi$ may be derived in a similar way,

$$
r = a: \quad \psi = -\delta U, \quad \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) - \frac{ac}{\varepsilon_0 dE_w} (E_{ow}) \frac{\partial \psi}{\partial r} = 0, \quad \tag{11}
$$

$$
r = b: \quad \psi = 0, \quad z = 0, H: \quad \frac{\partial \psi}{\partial z} = 0, \quad \tag{12}
$$

where $\delta U = U - U_0$.

Eqs. (10)–(12) represent a linear non-homogeneous boundary-value problem, the non-homogeneity being introduced by the term $\delta U$. Since this problem describes the vicinity of a bifurcation point, its solution must be non-unique. This means that the corresponding homogeneous problem also has a non-unique solution, i.e., is degenerate. One may attempt to find this solution by means of separation of variables, i.e., in the form $\psi = R(r) \Theta(\theta) Z(z)$. The functions $\Theta$ and $Z$ can be found easily,

$$
\Theta_m^{(1)} = \sin m \theta, \quad \Theta_m^{(2)} = \cos m \theta, \quad Z_n = \cos \frac{\pi n}{H} z, \quad m, n = 0, 1, 2, \ldots \tag{13}
$$

The function $R$ is governed by the problem

$$
\frac{d}{dr} \left[ E_0 \frac{d}{dr} \left( r \frac{dR}{dr} \right) + w \frac{dR}{dr} - \left( \frac{m^2}{r^2} + \frac{\pi^2 n^2}{H^2} \right) rE_0 R \right] - \left( \frac{m^2}{r^2} + \frac{\pi^2 n^2}{H^2} \right) wR = 0, \tag{14}
$$

$$
r = a: \quad R = 0, \quad \frac{d}{dr} \left( r \frac{dR}{dr} \right) - \frac{ac}{\varepsilon_0 dE_w} (E_{ow}) \frac{dR}{dr} = 0, \tag{15}
$$

$$
r = b: \quad R = 0.
$$

Eq. (14) together with boundary conditions (15) represent a linear homogeneous boundary-value problem for an ordinary differential equation. According to the above, this problem must be degenerate.

Thus, one should attempt to find values of $U_0$ for which the problem (14), (15) is degenerate. This was done numerically in the following way. The finite difference scheme of the fourth order of accuracy was applied. The determinant of the resulting system of linear algebraic equations was evaluated by means of the variant of Gauss’ exclusion technique. Such an evaluation was made for a given geometry of the discharge and for varying $U_0$. A change of sign of the determinant would indicate passing through a bifurcation point.

Calculations have been performed in the range of $U_0$ up to 200 kV. The geometrical parameters were chosen similar to those of Ref. [11]: the radius of the corona cathode, $a$, was taken equal to 3 or 15 mm, the radius of the grounded anode, $b$, was 15 cm, and the height of the discharge tube, $H$, was 15 cm. Parameters $m$ and $n$ varied independently from 0 to 4, excluding the variant $m = n = 0$ in which no multidimensional solutions branch off (i.e., bifurcation points were sought in which the first 24 harmonics branch off).

No change of the determinant sign was detected in the whole range of conditions studied. In other words, bifurcation points are absent, which indicates that multidimensional solutions probably do not exist.

4. Concluding remarks

The performed analysis shows that there are no bifurcation points on the one-dimensional steady-state cylindrically symmetric solution obtained in the framework of the model [11,15] of the corona discharge. This indicates that there are probably no regimes with current contraction in the framework of the considered model. One can conclude that the current-free strips on the corona cathode surface are due to the specific discharge geometry rather than the current contraction. The fact that the weak geometrical factor can produce a finite effect can probably be explained by the strong dependence of the current density at the corona onset on the local electric field.
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