Maxwell’s Construction for Non-linear Heat Structures and Determination of Radius of Arc Spots on Cathodes

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Abstract

The temperature distribution in a half-space (the cathode bulk) heated by an energy flux coming from the adjacent media (plasma) is considered for the case when the dependence of the energy flux density on the local surface temperature is localized in a narrow temperature range. A condition is derived of co-existence of a region that collects the energy flux with a surrounding energy flux-free region (Maxwell’s construction). This condition allows one to calculate the radii of current spots on arc cathodes.

1. Introduction

Calculation of current transfer to electrodes in arc discharges is a problem of scientific challenge and of technological interest which has been under intensive investigation for many years; see, e.g., reviews [1–8] and references therein. Most of the effort has been directed to cathodes; anode phenomena are usually believed to be less critical to the performance of arc devices. At present, the only approach capable of given quantitative results is the one considering processes in the near-cathode plasma as quasi-stationary. The only (if any) non-stationary process in the framework of this approach is heat propagation in the cathode bulk.

A self-consistent description of current transfer to cathodes in the framework of this approach may be obtained by means of numerical modelling with iterative adjustment of solutions in the near-cathode plasma and in the cathode body [9–14]. In accordance to the physical situation considered, the arc current is the only control parameter in this approach for a given cathode material and a given gas and its pressure, all the other quantities are calculated. In particular, the distribution of the temperature in the cathode body and on the surface and the voltage drop in the near-cathode plasma layer are obtained in the course of calculations.

On the other hand, a simple and physically transparent model can be obtained by postulating that the cathode surface can be divided into a circular current-collecting region (a spot) with a more or less distinct boundary and a surrounding current-free region. Such an approach has been widely employed and has given interesting and useful results (see, e.g., reviews [1–4, 6, 7] and references therein; we mention also recent studies [15, 16]). Unfortunately, this approach is still incomplete. The above-mentioned recent works represent a typical example: the voltage drop in the near-cathode plasma layer is considered in [16] as an empirical parameter instead of being calculated; in [15], the “principle” of minimum voltage is invoked, which does not follow from the governing equations (see in this regard a discussion of application of this principle to the cylindrical arc [17–19]).

A reason of the incompleteness is as follows. The temperature of the cathode surface within the spot is assumed to be constant and governed by an equation of integral heat balance of the cathode surface within the spot. A term of this equation accounting for the heat power removed from the spot by heat conduction into the cathode body is calculated by means of a two-dimensional heat conduction equation in the cathode body.) Thus, the condition of equality of the density of the energy flux coming to the cathode surface from the plasma and of the density of the heat flux removed from the surface by heat conduction into the cathode body, which holds at each point of the surface, is replaced by only one condition (equation of integral heat balance), which is insufficient to determine two parameters of the model (the temperature within the spot and the spot radius). Just this loss of information results in the incompleteness of the model: one parameter remains indeterminate.

In order to obtain a self-consistent description of the spot, one should derive a model from the governing equations rather than to just postulate it. Such an approach was developed in [12]. The radius of the spot is determined in the framework of this approach by a condition of solvability of the problem that governs the temperature distribution in the vicinity of the spot edge. However, an explicit form of this condition was not found in [12]. A particular form of this condition was derived for a specific case when the dependence of the density of the energy flux coming from the plasma to the cathode surface on the local surface temperature is described by a rectangular function [20].

It is the aim of this brief contribution to deduce an explicit form of the above-mentioned condition for a general case. This will allow one to treat more realistic dependencies of the density of the energy flux coming from the plasma to the cathode surface on the local surface temperature in a reasonable agreement with the experiment.

Being important for practical purposes, the topic in question is also of substantial theoretical interest due to its connection to the theory of nonlinear dissipative structures. From the point of view of this theory, the problem of contact of a spot with a surrounding current-free region is a problem of co-existence of phases. Generally, the co-existence is possible only if a certain condition is satisfied (Maxwell’s construction; see, e.g., [22]), which follows from a treatment of an intermediate (transition) region that
separates the phases. Evidently, the above-mentioned solvability condition represents Maxwell’s construction for nonlinear heat structures. From this point of view, the considered problem represents an example of a multidimensional model in which Maxwell’s construction can be formulated explicitly.

2. The model

A statement of the problem is the same as that in [12]. It is illustrated by Fig. 1 and may be briefly described as follows. If the dimensions and the radius of curvature of the current-collecting surface of the cathode are much larger than the spot radius, the bulk of the cathode may be considered as an infinite half-space while treating the spot. The surface of the half-space is heated by the energy flux coming from the adjacent plasma. We are interested in the temperature distribution created by this energy flux in the half-space. At large distances from the spot, this distribution is uniform: $T = T_0$, where $T_0$ is a given constant.

We introduce the cylindrical coordinates $(r, \varphi, z)$ with the polar $z$-axis directed from the cathode surface into the cathode bulk and will seek stationary and axially symmetric distributions: $T = T(r, z)$. Neglecting the Ohmic heating in the body of the cathode, one can write the equation of heat conduction in the half-space in the form of the two-dimensional Laplace equation

$$ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} = 0, $$

(1)

where $\psi$ is the so-called heat flux potential related to the thermal conductivity $\kappa$ of the cathode material by the formula

$$ \psi(T) = \int_{T_0}^T \kappa(T) \, dT. $$

(2)

Boundary conditions read

$$ z = 0: \quad \frac{\partial \psi}{\partial z} = -q(\psi, U), $$

(3)

$$ r + z \to \infty: \quad \psi \to 0, $$

(4)

where $q$ is the density of the energy flux coming from the plasma which will be considered as a known function of the local surface temperature (or, which is equivalent, of the respective value of $\psi$) and of the voltage drop in the near-cathode layer $U$, which is assumed to be the same at all points of the surface.

The above-described approach is applicable provided that the radius of the spot exceeds essentially the thickness of the near-cathode plasma layer in which the energy flux to the cathode surface is formed, then the current transfer through the near-cathode layer is locally one-dimensional and governed by the local surface temperature and by the voltage drop in the near-cathode layer. The structure of the near-cathode plasma region is depicted in Fig. 1. According to conventional concepts, the energy flux is formed, to a first approximation, in the space-charge sheath and in the ionization layer. The thickness of the space-charge sheath is of the order of the Debye length, the thickness of the ionization layer is of the order of the ionization length. Thus, a condition of validity of the considered approach is that the Debye and ionization lengths be much smaller than the spot radius.

As an example, we consider the atmospheric-pressure argon plasma, which is a frequent object of arc experiments. Assuming that the electron and heavy-particle temperatures are equal to $2 \cdot 10^4 \, K$ and calculating the ionization length by means of eq. (10) of the work [23] with the ionization rate constant for argon estimated according to Sec. VII of [23], one finds that the Debye and ionization lengths are approximately 0.02 μm and 50 μm, respectively. The spot radius for arcs with currents of the order of $10^4 \, A$ is usually believed to be of the order of 1 mm. Thus, the above condition is satisfied.

3. Asymptotic treatment

Examples of the dependence of $q$ on the surface temperature for the near-cathode layers in vacuum arcs and in arcs in ambient gas are given in [12, 21], respectively. In both cases, this function is non-zero only in a narrow range of the temperature values, which is a consequence of the fact that some of the processes involved are of the Arrhenius character with a high activation energy. It is natural to make use of the smallness of the width of this range and employ the method of matched asymptotic expansions [24–28] in order to get an approximate analytic solution of the problem. Such an analysis is presented in this section. For brevity, a small parameter and dimensionless variables are not introduced explicitly; a more formal treatment can be found in [12].

Designate by $T_0$ a value of the surface temperature inside the above range; for example, it can be the value in the point of maximum of the function $q(T)$ (for a given $U$). It is evident that the temperature of the cathode surface within the spot cannot substantially exceed $T_0$, otherwise the local heat flux density would be zero and this temperature would not be maintained. Thus, one can write to a first approximation

$$ r < r_\ast: \quad \psi(r, 0) = \psi_\ast, $$

(5)

where $r_\ast$ is the radius of the spot and $\psi_\ast = \psi(T_0)$ is the value of the heat flux potential that corresponds to the point of maximum of $q(T)$. The temperature of the cathode surface outside the spot is below the above-mentioned range and the density of the energy flux is zero:

$$ r > r_\ast: \quad \frac{\partial \psi}{\partial z} (r, 0) = 0. $$

(6)

A solution to the Laplace equation (1) subject to these boundary conditions and to the boundary condition (4) reads

$$ \psi = \frac{2\psi_\ast}{\pi} \arcsin \left[ \frac{2r_\ast}{\sqrt{(r - r_\ast)^2 + z^2}} \right]^{1/2} + \frac{2r_\ast}{\pi} \sqrt{(r + r_\ast)^2 + z^2} \left[ \frac{2r_\ast}{\sqrt{(r + r_\ast)^2 + z^2}} \right]^{1/2}. $$

(7)
The density of the energy flux coming from the plasma to the surface occupied by the spot and the integral energy flux derived from this solution are, respectively

\[ q(r) = \frac{2\psi_*}{\pi} \frac{1}{\sqrt{r_*^2 - r^2}} Q = 4r_* \psi_* . \]  

(8)

The energy flux density has an (integrable) singularity at \( r = r_* \). Hence, the solution (7) is not applicable in the vicinity of the ring \( r = r_* \) in the plane \( z = 0 \), i.e., in the vicinity of the spot edge.

The temperature distribution in the above vicinity is governed, to a first approximation, by the equation

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0, \]  

(9)

where \( x = r - r_* \).

One of boundary conditions for eq. (9) is supplied by eq. (3). Another condition follows from the principle of asymptotic matching and may be obtained by transforming eq. (7) to variables \( \rho, \phi \) (which are the polar coordinates in the plane \( x, z : x = \rho \cos \phi, z = \rho \sin \phi \)), expanding in powers of \( \rho \) and retaining two terms:

\[ \psi = \psi_* - \left( \frac{8\rho}{\pi r_*^2} \right)^{1/2} \cos \frac{\phi}{2}. \]  

(10)

This condition applies at \( \rho \) much larger than the length scale in the vicinity of the spot edge (see the estimate in [12]) however much smaller than the spot radius.

4. Maxwell's construction

The problem (9), (10), and (3) was formulated in [12]. It represented, to a first approximation, by the equation

\[ \int_0^R \left[ \frac{\partial \psi}{\partial x} \right]^2 x = \sqrt{r_*^2 - z^2} \]  

(11)

Complete the integration over \( x \) in the first term and integrate by parts in \( z \) in the second term

\[ \frac{1}{2} \int_0^R \left[ \frac{\partial \psi}{\partial x} \right]^2 x = \sqrt{r_*^2 - z^2} \]  

(12)

The third term, after use has been made of the boundary condition (3), may be transformed to the integration variable \( \psi \). The fourth term, after the integration in \( x \) has been completed, may be combined with the first term. One gets

\[ \int_0^R \left[ \frac{\partial \psi}{\partial x} \right]^2 x = \sqrt{r_*^2 - z^2} \]  

(13)

It is convenient to transform the left-hand side to variables \( \rho, \phi \):

\[ \int_0^R \left[ \frac{\partial \psi}{\partial \rho} \right]^2 - \sin \phi \frac{\partial \psi}{\partial \rho} \frac{\partial \psi}{\partial \phi} - \cos \phi \left( \frac{\partial \psi}{\partial \phi} \right)^2 \right] \]  

(14)

We choose \( R \) in such a way that it exceeds substantially the length scale in the considered vicinity however is much smaller than the radius of the spot. Then the left-hand side may be evaluated by means of the boundary condition (10). Values \( \psi(-R, 0) \) and \( \psi(R, 0) \) for such \( R \) are outside the range in which the function \( q \) is localized, hence the interval of integration on the right-hand side includes the whole of this range. For definiteness, one may set the limits of integration equal to zero and infinity. One gets

\[ \frac{\psi_*^2}{\pi r_*} = \int_0^\infty q(\psi, U) \]  

(15)

In contrast to all the preceding equations of this section, eq. (15) does not involve a solution to the problem (9), (10), and (3), i.e., the function \( \psi \), but rather involves only the input parameters of this problem. Hence, this equation represents a condition of solvability of the problem (9), (10), and (3).

One can say that eq. (15) establishes a condition of coexistence of the hot and cold phases (the spot and the surrounding current-free region). From this point of view, eq. (15) may be called Maxwell's construction for the heat structures considered. It should be emphasized that eq. (15) is not just a trivial consequence of continuity of the heat flux.

5. Concluding remarks

In the framework of the model considered, the temperature of the cathode surface within the spot is to a first approximation equal to \( T_{\text{th}} \), that is, to the value in the point of maximum of the function \( q(T) \) for fixed \( U \). The radius of the spot is governed by eq. (15), which gives, after the heat flux potential has been excluded

\[ r_* = \frac{1}{\pi} \int_{T_o}^{T_*} \kappa(T) dT \int_{T_o}^\infty q(T, U) \kappa(T) dT \]  

(16)

Thus, if the functions \( \kappa(T) \) and \( q(T, U) \) are specified (we remind that these functions describe dependencies of, respectively, thermal conductivity of the cathode material on the temperature and of the density of the energy flux from the plasma on the local temperature of the cathode surface and on the voltage drop in the near-cathode layer), the spot parameters may be uniquely determined. It should
be emphasized that eq. (16), governing the spot radius, follows from analysis of the transition zone separating the spot and the surrounding current-free region, which is typical for such co-existence conditions.

It may happen in a particular experimental situation that the spot radius is not much smaller that the dimensions of the cathode but rather is comparable to these dimensions. In such cases, the outer solution (7) is no longer valid. On the other hand, the analysis of Section 4 refers to a thin zone and therefore remains applicable, with the exception of eq. (15) which may be easily generalized.

The present treatment has been carried out for the case when the function $q(T, U)$ for a given $U$ is localized in a narrow range of surface temperatures. Such a requirement is in fact quite natural: it is a necessary condition for the model of a spot with a constant temperature and a distinct boundary to be justified. (If this condition is not satisfied, the temperature of the cathode surface within the spot is not constant and, additionally, a more or less distinct boundary between the current-collating and current-free regions may not exist, in which case no definite spot radius can be introduced; examples can be found in [12].) As it has been pointed out above, available calculations of the near-cathode layers in arc discharges [12, 21] result in functions $q$ of just this type.

The present results can be compared with those of the previous works [12, 20], in which calculations for specific functions $q$ have been performed. Consider first the case [20] when the dependence $q(\psi)$ for fixed $U$ is described by a rectangular function

$$q(\psi) = \begin{cases} q_*, & \text{if } \psi < \psi_*(1 - \varepsilon) \\ q_*, & \text{if } \psi > \psi_*(1 + \varepsilon) \\ 0, & \text{if } \psi < \psi_*(1 - \varepsilon) < \psi < \psi_*(1 + \varepsilon), \end{cases} \quad (17)$$

where $q_*, \psi_*$, and $\varepsilon$ are fixed, $\varepsilon$ being a small parameter. Equation (15) gives $r_* = \psi_*/2n\varepsilon q_*$, which conforms the respective result [20].

Numerical calculations [12] were performed for the function

$$q = q_* \left( \cosh \frac{\psi - \psi_*}{\varepsilon \psi} \right)^{-1}. \quad (18)$$

As above, $q_*, \psi_*$, and $\varepsilon$ are fixed in this expression, $\varepsilon$ being a small parameter. The obtained values of the dimensionless spot radius and of the dimensionless integral heat flux are shown in Table I.

The function (18) is localized in the range $|\psi - \psi_*| = O(\varepsilon \psi_*)$. In this range, the quantity $\psi$ in the denominator may be replaced, to a first approximation, with $\psi_*$. The lower limit of integration on the right-hand side of eq. (15) may be set equal to minus infinity, to an accuracy of exponentially small terms. Evaluating the integral on the right-hand side of eq. (15) with account of these simplifications, one gets the value $\pi \varepsilon q_* \psi_*$. The spot radius and the integral heat flux are

$$r_* = \frac{\psi_*}{\pi \varepsilon q_*}, \quad Q = \frac{4\psi_*^2}{\pi^2 \varepsilon q_*}. \quad (19)$$

Evaluation by means of these formulas gives values lower than those given by the numerical calculation; the respective relative differences are shown in the parentheses in Table I. As it should have been expected, the difference between the asymptotic and numerical values decreases with decreasing $\varepsilon$, i.e., with decreasing width of the range of localization of the function $q$.

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References


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Table I. Numerical results and comparison of the asymptotic and numerical results for the model (18)

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$2r_<em>/q_</em>\psi_*$</th>
<th>$Q\psi_*^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.04 (38%)</td>
<td>7.35 (45%)</td>
</tr>
<tr>
<td>0.05</td>
<td>1.76 (27%)</td>
<td>11.9 (32%)</td>
</tr>
</tbody>
</table>