I. INTRODUCTION

The problem of a transition from a collision-dominated plasma via a quasineutral Knudsen layer and a collision-free space-charge sheath to a negative wall has been considered previously under the assumption of zero ion temperature. This assumption amounts to supposing that the ion temperature is much smaller than the electron temperature and thus the ion pressure term in the ion momentum equation may be neglected as compared with the force applied by the electric field (which is governed by the electron temperature).

The above-mentioned problem has a number of features in common with the problem of distribution of parameters in a cross section of a low-pressure glow discharge column which was addressed, e.g., in Refs. 2–6. In Ref. 2, the assumption of zero ion temperature was used. In Refs. 3–5, the ion temperature was assumed to be spatially uniform and either small, or small or finite, or finite. In Ref. 6, the ion temperature was assumed to be proportional to the squared mean ion velocity.

The above-cited works are based on the fluid description of the ion motion. In order to further improve the model, it seems important to address the question of ion temperature. Indeed, none of the above assumptions is obligatory in the framework of the fluid model; one can include in the analysis the ion energy equation and consider a distribution of the ion temperature as an additional unknown function. Note that one of the mechanisms that cause a deviation of the ion temperature from the (spatially uniform) atom temperature is heating due to conversion of kinetic energy of the ion flow into thermal energy in collisions of ions with atoms; in connection with this mechanism, a question arises regarding the applicability of the assumption of zero ion temperature; even if the temperature of the atoms is much smaller than the electron temperature, the ion temperature may exceed substantially the temperature of atoms and the question as to whether it is much lower than the electron temperature requires a special investigation.

In this work, the problem of plasma–wall transition is stated in the framework of the fluid model with account of variability of the ion temperature and solved asymptotically in the limit case of small ratio of the Debye length to the characteristic mean free path for collisions ion-neutral. The problem is formulated in Sec. II. In Sec. III, the ion energy equation is qualitatively analyzed by means of simple estimates. The problem is transformed to dimensionless variables in Sec. IV. Asymptotic analysis is given in Sec. V. Results are presented and discussed in Sec. VI. In Sec. VII, the solution obtained for the space-charge sheath is compared with the free-fall solution.

II. EQUATIONS AND BOUNDARY CONDITIONS

Consider an atomic plasma with the ionization degree much less than unity at rest near an absorbing surface under a negative potential. The ion fluid is described by equations of conservation of number, momentum and energy

\[ n_i \frac{d \mathbf{v}_i}{dy} = \frac{d(n_i kT_i)}{dy} - en_i \frac{d \phi}{dy} + \frac{en_i}{\mu_i} \mathbf{v}_i, \]

\[ m_i n_i \frac{d \mathbf{v}_i}{dy} = \frac{d(n_i kT_i)}{dy} + \frac{3}{2} \zeta_{ia} e n_i k(T_i - T_a) \frac{2 \mu_i m_i}{(T_i + T_a)} - \frac{en_i T_i \mathbf{v}_i^2}{\mu_i (T_i + T_a)}. \]

Here the y-axis is directed from the surface into the plasma, \( n_i \) is the number density of ions, \( \mathbf{v}_i \) is the mean velocity of the motion of ions in the direction to the surface, \( J_j \) is the density of the ion flux to the surface (a given constant), \( T_j \) is the ion temperature, \( \phi \) is the electrostatic potential, \( \mu_i \) is the ion mobility, \( \zeta_{ia} \) is a dimensionless coefficient which is expressed in terms of integrals of the weighted cross section for momentum transfer in ion–atom collisions (see Ref. 7), \( T_a \) is the temperature of atoms (a given constant), \( m_i \) is the mass of the ions.
of ion. The equations are written under the assumption that volume ionization and recombination are not essential on the length scale considered. The viscous stress term in the momentum equation is neglected, as is usual in the fluid model (see, e.g., Ref. 3). Also neglected is the heat conduction term in the energy equation.

The term on the left-hand side of Eq. (2) accounts for the ion inertia, the terms on the right-hand side account for, respectively, ion pressure gradient, electric field, and the friction force. The term on the left-hand side of Eq. (3) describes the material derivative of ion internal energy. The first term on the right-hand side accounts for work of the pressure force. The second and third terms on the right-hand side of Eq. (3) account for change of energy of ions due to their elastic collisions with atoms. The form of these terms follows from Ref. 7 if one applies the results of Ref. 7 to ion–atom collisions in an atomic plasma with the ionization degree much less than unity, taking into account that the mean velocity of atoms in such a plasma is much smaller than the ion velocity and that masses of an ion and of an atom in an atomic plasma are nearly equal; note that the ion mobility is much less than unity, taking into account that the mean velocity of atoms in such a plasma is much smaller than the ion velocity and that masses of an ion and of an atom in an atomic plasma are nearly equal; note that the ion mobility is related to the coefficient $D_{ia}$ used in Ref. 7 by the formula $\mu_{ia}=eD_{ia}/k(T_a+T_i)$. The second term describes the exchange of thermal energy between ions and atoms, the third term describes heating of ions due to conversion of kinetic energy of their directed motion into thermal energy in elastic collisions with atoms.

Assuming that the neutral gas pressure and temperature are given, we consider the ion mobility $\mu_i$ and the coefficient $\xi_{ia}$ as known functions of $T_i$ and $\nu_i$; $\mu_i=\mu_i(T_i,\nu_i)$, $\xi_{ia}=\xi_{ia}(T_i,\nu_i)$. Note that the quantity $\xi_{ia}$ equals unity for $\nu_i$ much smaller than thermal velocities.

The transport equation for electrons is written under the usual assumption that the electron temperature is spatially uniform and that electron inertia and friction force are negligible,

$$-kT_e\frac{dn_e}{dy}+en_e\frac{d\phi}{dy}=0.$$  

(4)

Here $n_e$ is the electron number density and $T_e$ is the electron temperature (which will be considered as a given constant).

The system is closed by the Poisson equation

$$\varepsilon_0\frac{d^2\phi}{dy^2}=-e(n_i-n_e).$$  

(5)

Boundary conditions at $y\to\infty$ are those of asymptotic matching with a solution given by the diffusion theory under the assumption of quasineutrality

$$n_i=Ay+\cdots,\quad n_e=Ay+\cdots,$$  

(6)

$$\nu_i=\frac{J_i}{A}y+\cdots,\quad T_i\to T_a,\quad \phi=\frac{kT_e}{e}\left[\ln y/B+O(1)\right],$$  

(7)

where

$$A=\frac{eJ_i}{k(T_a+T_e)\mu_{i\infty}},$$  

(8)

$\mu_{i\infty}=\mu_i(T_a,0)$, and $B$ is a given quantity having dimension of a distance. One can check easily that the right-hand sides of Eqs. (6) and (7) satisfy the system of Eqs. (1)–(5).

A boundary condition at $y=0$ is obtained by specifying the local potential

$$\phi=\phi_w,$$  

(9)

where $\phi_w$ is the potential of the surface (a given quantity).

After the above-stated problem has been solved, one gets a complete information about the distribution across the near-wall region of the potential, electric field, ion and electron densities, mean ion velocity. In particular, one can determine the electric field at the surface, which governs the electron emission from the surface, and the energy transported to the surface by ions. In order to determine the latter, the use will be made of the conventional hydrodynamics formula

$$q=J_i\left(\frac{5kT_{iw}}{2}+\frac{m_i\nu^2_{iw}}{2}\right),$$  

(10)

where $q$ is the density of the ion energy flux to the surface and the index $w$ here and further is attributed to values of respective quantities at the surface. The first term in the parentheses on the right-hand side of Eq. (10) is the enthalpy per ion and the second term is the kinetic energy of the directed motion.

III. ESTIMATES

Before proceeding to a mathematical treatment, it is advisable to analyze the ion energy equation qualitatively by means of simple order-of-magnitude estimates. The order of magnitude of the electrostatic potential may be determined from Eq. (4), $\phi=O(kT_e/e)$. Assuming that the ion inertia term of Eq. (2) is comparable with the field term, one finds $\nu_i=O(\sqrt{kT_e/m_i})$. Assuming that the ion inertia term and the friction term are comparable (which implies that the plasma is neither collision-dominated nor collision-free; in other words, the estimates apply to the Knudsen layer), one finds that the characteristic length scale is of the order of $\sqrt{kT_e/m_i\mu_i/e}$. Estimating terms of Eq. (3) under the natural assumption $T_i\geq O(T_a)$, one finds that orders of magnitude of the terms on the right-hand side relative to the term on the left-hand side are 1, $T_e/T_i$, respectively. Since the third term on the right-hand side cannot be the only one dominating in the equation, one should assume that $T_e/T_i\ll O(1)$.

It follows that $T_i$ in the bulk of the Knudsen layer cannot be asymptotically small as compared to $T_e$, without regard of whether $T_a$ is or not asymptotically small. The reason is the presence of the last term on the right-hand side of the ion energy equation, which describes heating of ions due to conversion of kinetic energy of their directed motion into thermal energy in elastic collisions of ions with atoms. In other words, the hypersonic regime of ion flow (which is the same as the regime of asymptotically cold ions) cannot be realized in a medium with scattering centers. Since ions arrive at the edge of the space-charge sheath after having been heated in the Knudsen layer, the regime of asymptotically cold ions cannot be realized in the sheath as well.
IV. TRANSFORMING THE PROBLEM TO DIMENSIONLESS VARIABLES

Equation (4) may be integrated to give the Boltzmann distribution

\[ n_e = n_0 \exp \left( \frac{e \phi}{kT_e} \right), \]

(11)

where \( n_0 = AB \).

We exclude \( n_i \) and \( n_e \) from Eqs. (2), (3), and (5) by means of Eqs. (1) and (11) and introduce dimensionless variables

\[ \eta = \frac{y}{\lambda}, \quad V = \frac{v_i}{u_i}, \quad \theta = \frac{T_i}{T_a + T_e}, \]

(12)

where \( u_i = \sqrt{k(T_a + T_e)} / m_i \) is a characteristic velocity of ions, \( \mu_i^{(0)} \) is a characteristic value of the ion mobility [for example, one can set \( \mu_i^{(0)} = \mu_i(T_a + T_e, 0) \)], and \( \lambda = u_i m_i \mu_i^{(0)} / e \) is a characteristic mean free path for collisions ion–atom. The system of equations assumes the form

\[ (\theta - V^2) \frac{dV}{d\eta} = V \frac{d\theta}{d\eta} + VE - \frac{V^2}{b}, \]

(13)

\[ \frac{3}{2} V \frac{d\theta}{d\eta} = - \theta \frac{dV}{d\eta} + \frac{3 \zeta_{ia}}{2b} (\theta - \gamma) - \frac{\theta V^2}{b(\theta + \gamma)}, \]

(14)

\[ \varepsilon V \frac{d\varepsilon}{d\eta} = e^{\Phi} - 1, \quad (1 - \gamma) \left( \frac{d\Phi}{d\eta} - \frac{1}{V} \frac{dV}{d\eta} \right) = E, \]

(15)

where

\[ \gamma = \frac{T_a}{T_a + T_e}, \quad \varepsilon = (\frac{h}{\lambda})^2, \]

\[ h = \left( \frac{e_0 k(T_a + T_e) u_i}{\varepsilon^2 \mu_i} \right)^{1/2}, \quad b = \frac{\mu_i^{(0)}}{\mu_i}. \]

(16)

Note that the function \( \Phi \) may be considered as a part of the electrostatic potential that is responsible for appearance of the space charge; if \( \Phi = 0 \), \( n_e = n_i \) and no space charge is present, which may be conveniently seen from the first equation in Eq. (15). The ratio of the ion flux density \( J_i \) to the characteristic velocity \( u_i \) represents a characteristic charged particle density, hence the parameter \( h \) has the meaning of a characteristic Debye length.

The boundary conditions (7) and (9) assume the form

\[ \eta \to \infty: \quad V = \frac{b}{\lambda} + \cdots, \quad \theta \to \gamma, \]

(17)

\[ E = \frac{1 - \gamma}{\eta} + \cdots, \quad \Phi \to 0; \]

(18)

\[ \eta = 0: \quad \Phi - \ln V = \chi, \]

where the parameter \( \chi \) is defined by the formula

\[ \chi = \frac{e \phi_w}{kT_e} - \ln \frac{\lambda b}{B}, \]

(19)

and has the meaning of a dimensionless voltage drop in the region considered.

The electric field at the surface and the energy transported to the surface by ions are expressed in terms of the dimensionless quantities as follows:

\[ \frac{d\Phi}{d\eta} = \frac{u_i}{\mu_i^{(0)}} E_w, \quad q = J_i k(T_a + T_e) \left[ \frac{5}{2} \theta + \frac{V^2}{2} \right]. \]

(20)

The formulated problem is governed by two parameters; \( \gamma \) characterizing the ratio of the electron and atom temperatures and the squared ratio \( \varepsilon \) of the characteristic Debye length to the characteristic mean free path. In the case of cold atoms, \( T_a / T_e \to 0 \), \( \gamma \) tends to zero; \( \gamma \) equals one-half in the case when the plasma is in thermal equilibrium at the edge of the region considered. Note that no solution with \( \theta = O(\gamma) \) exists in the limit \( \gamma \to 0 \), which is due to the presence of the last term on the right-hand side of the energy equation (14) and conforms to the conclusion drawn in the preceding section.

Parameter \( \varepsilon \) may vary in a wide range. In this work, an asymptotic solution of the formulated problem in the limit \( \varepsilon \to 0 \) is obtained.

Before proceeding to the asymptotic treatment, it is convenient to write down the following equation which may be derived from Eq. (13) by excluding \( d \theta / d \eta \) by means of Eq. (14) and \( E \) by means of the second equation in Eq. (15).

\[ \left( \frac{5}{3} \theta - V^2 + 1 - \gamma \right) \frac{dV}{d\eta} = (1 - \gamma) V \frac{d\Phi}{d\eta} + \frac{\zeta_{ia}}{b} (\theta - \gamma) - \frac{V^2 (5 \theta + 3 \gamma)}{3b(\theta + \gamma)}. \]

(21)

V. ASYMPTOTIC TREATMENT

In this section, an asymptotic solution of the considered problem in the limit \( \varepsilon \to 0 \) is obtained by means of the method of matched asymptotic expansions (see, e.g., Refs. 8–12). A physical meaning of the obtained asymptotic results is discussed and numerical solution of the derived equations is given in the next section.

Asymptotic structure of the solution is similar to that found earlier under the assumption of cold ions (see, e.g., Ref. 1, and references therein). Three asymptotic expansions should be considered in order to completely describe the solution. A straightforward asymptotic expansion is applicable at finite positive \( \eta \) and reads

\[ V = V_i(\eta) + \cdots, \quad \theta = \theta_i(\eta) + \cdots, \]

\[ E = E_i(\eta) + \cdots, \quad \Phi = \varepsilon \Phi_i(\eta) + \cdots. \]

(22)

Substituting this expansion into Eqs. (13)–(15) and retaining leading terms, one can see that Eqs. (13) and (14) do not change their appearance (\( V, \theta, \) and \( E \) being replaced by the respective quantities with index 1); Eqs. (15) assume the form

\[ \Phi_i = V_i \frac{dV_1}{d\eta}, \quad E_i = \frac{1 - \gamma}{V_1} \frac{dV_1}{d\eta}. \]

(23)
Excluding $E_1$ from Eq. (13) by means of the second equation in Eq. (23) and resolving the obtained equation jointly with Eq. (14) with respect to derivatives, one arrives at the following system of equations for functions $V_1(\eta)$ and $\theta_1(\eta)$ [note that the first one of the following equations may be obtained in a more direct way by expanding Eq. (21)]:

\[
\frac{5}{3} \theta_1 - V_1^2 + 1 + \gamma \frac{dV_1}{d\eta} = \frac{\xi_{ia}}{b}(\theta_1 - \gamma) - \frac{V_1^2(5 \theta_1 + 3 \gamma)}{3 b(\theta_1 + \gamma)},
\]

\[
(24)
\]

\[
\frac{5}{3} \theta_1 - V_1^2 + 1 + \gamma \frac{d\theta_1}{d\eta} = \frac{\xi_{ia}}{bV_1}(\theta_1 - \gamma)(\theta_1 - V_1^2 + 1 - \gamma) + \frac{2 \theta_1 V_1}{3 b(\theta_1 + \gamma)}(V_1^2 + 2 \gamma - 1).
\]

\[
(25)
\]

These equations are subject to boundary conditions

\[
\eta \to \infty: \quad V_1 = \frac{b}{\eta} + \cdots, \quad \theta_1 \to \gamma;
\]

\[
\eta = 0: \quad V_1^2 - \frac{2}{3} \theta_1 = 1 - \gamma.
\]

The last boundary condition requires that the singularity in Eqs. (24) and (25) occur (i.e., the coefficient in front of the derivatives vanish) at the boundary.

A numerical solution of problem (24)–(26) is given in the next section. After this problem has been solved, one can determine functions $E_1(\eta)$ and $\Phi_1(\eta)$ by means of Eqs. (23).

The next asymptotic expansion to be considered is related to the limit $\varepsilon \to 0$, $\eta_2 = \eta/\varepsilon^{2.5}$ finite and positive and reads

\[
V = V_s + \varepsilon^{1.5} V_2(\eta_2) + \cdots, \quad \theta = \theta_s + \varepsilon^{1.5} \theta_2(\eta_2) + \cdots,
\]

\[
E = \varepsilon^{0.5} E_2(\eta_2) + \cdots, \quad \Phi = \varepsilon^{1.5} \Phi_2(\eta_2) + \cdots
\]

where $V_s = V_1(0)$ and $\theta_s = \theta_1(0)$.

Equations (13)–(15) to a first approximation assume the form

\[
(\theta_s - V_s^2) \frac{dV_s}{d\eta_2} = V_s \frac{d\theta_s}{d\eta_2} + V_s E_2,
\]

\[
(28)
\]

\[
\frac{3}{2} \frac{d\theta_s}{d\eta_2} = -V_s \frac{dV_s}{d\eta_2},
\]

\[
(29)
\]

\[
\Phi_2 = V_s \frac{dE_2}{d\eta_2}, \quad E_2 = -\frac{1 - \gamma}{V_s} \frac{dV_s}{d\eta_2}.
\]

\[
(30)
\]

Excluding $E_2$ from Eq. (28) by means of the second equation in Eq. (30), one arrives at an equation coinciding, to the accuracy of a factor, with Eq. (29). Hence, one more relationship is necessary. This relationship may be obtained by considering a higher order approximation, however a simpler way is to employ Eq. (21), which to a first approximation assumes the form

\[
\frac{5}{3} \theta_2 - V_s V_2 \frac{dV_2}{d\eta_2} = (1 - \gamma) V_s \frac{d\Phi_2}{d\eta_2} - c_1.
\]

\[
(31)
\]

Here

\[
c_1 = -\frac{\xi_{ia}}{b_s} \left( \theta_s - \gamma \right) + \frac{V_s^2(5 \theta_s + 3 \gamma)}{3 b_s(\theta_s + \gamma)},
\]

\[
(32)
\]

where $b_s$ and $\xi_{ia}$ designate respective quantities evaluated at $\eta = \theta_s$, $V = V_s$.

Integrating Eq. (29), one arrives at the relationship

\[
\frac{3}{2} V_s \theta_s + \theta_2 V_2 = C_1,
\]

\[
(33)
\]

where $C_1$ is an integration constants.

Excluding $\theta_1$ from Eq. (31) by means of Eq. (33), integrating once and excluding $\Phi_2$ by means of Eqs. (30), one obtains an equation for the function $V_2$,

\[
(1 - \gamma)^2 V_s \frac{d^2V_2}{d\eta_2^2} - \frac{5}{9} \theta_s V_s (\theta_V V_2 - C_1)^2 - V_s V_2^2 = -c_1 \eta_2 + C_2,
\]

\[
(34)
\]

where $C_2$ is another integration constants.

Boundary conditions for this equation are

\[
\eta_2 \to \infty: \quad V_2 = -\sqrt{\frac{9c_1 V_s}{5 \theta_s + 9 V_s^2}} \eta_2 + \cdots,
\]

\[
(35)
\]

\[
\eta_2 \to 0: \quad V_2 = 54 V_s^2 (1 - \gamma^2)^2 \frac{1}{5 \theta_s + 9 V_s^2} \eta_2 + \cdots.
\]

One needs to consider second-order terms of the straightforward expansion in order to find the integration constants $C_1$ and $C_2$. After these constants have been determined, problem (34)–(35) may be solved, after which functions $\theta_2(\eta_2)$, $E_2(\eta_2)$, and $\Phi_2(\eta_2)$ can be found by means of Eqs. (33) and (30).

The last asymptotic expansion to be considered is related to the limit $\varepsilon \to 0$, $\eta_3 = \eta/\varepsilon^{1.5}$ finite and non-negative and reads

\[
V = V_3(\eta_3) + \cdots, \quad \theta = \theta_3(\eta_3) + \cdots,
\]

\[
E = \varepsilon^{-1.5} E_3(\eta_3) + \cdots, \quad \Phi = \varepsilon^{0.5} \Phi_3(\eta_3) + \cdots
\]

\[
(36)
\]

Equations (13)–(15) to a first approximation assume the form

\[
(\theta_3 - V_3^2) \frac{dV_3}{d\eta_3} = V_3 \frac{d\theta_3}{d\eta_3} + V_3 E_3,
\]

\[
(37)
\]

\[
\frac{3}{2} V_3 \frac{d\theta_3}{d\eta_3} = -\theta_3 \frac{dV_3}{d\eta_3},
\]

\[
(38)
\]

\[
V_3 \frac{dE_3}{d\eta_3} = e^{\Phi_3} - 1, \quad (1 - \gamma) \left( \frac{d\Phi_3}{d\eta_3} - \frac{1}{V_3} \frac{dV_3}{d\eta_3} \right) = E_3.
\]

\[
(39)
\]

Boundary conditions for these equations are

\[
\eta_3 = 0: \quad \Phi_3 - \ln V_3 = \chi,
\]

\[
(40)
\]

\[
\eta_3 \to \infty: \quad V_3 \to V_s, \quad \theta_3 \to \theta_s, \quad E_3 \to 0, \quad \Phi_3 \to 0.
\]

\[
(41)
\]

Equation (38) may be readily integrated to give
\[ \theta_3 = \theta_1 \left( \frac{V_3}{V_5} \right)^{2/3} \].

Excluding by means of this relationship \( \theta_3 \) from Eq. (37), one gets

\[ \frac{5}{3} \theta_3 \frac{V_3^{2/3}}{V_5} \frac{dV_3}{\frac{dV_3}{V_3}} = E_3. \] (43)

Excluding \( E_3 \) by means of the second equation in Eq. (39) and integrating, one can express \( \Phi_3 \) in terms of \( V_3 \),

\[ \Phi_3 = \frac{5}{2} \theta_3 \left( 1 - \frac{V_3}{V_5} \right) \left[ 1 - \frac{5}{3} \frac{V_3^{2/3}}{V_5^{5/3}} \right] \] (44)

Applying this equation at \( \eta_3 = 0 \), one can find a relation between \( \chi \) and the mean ion velocity at the surface.

Making use of the first equation in Eq. (39) and of Eq. (43), one can express \( E_3 \) in terms of \( V_3 \),

\[ \frac{E_3^2}{2} = \int V_3 \left( 1 - e^{\Phi_3} \right) \left[ 1 - \frac{5}{3} \frac{V_3^{2/3}}{V_5^{5/3}} \right] dV_3. \] (45)

In order to evaluate the integral, it is convenient to represent it as a sum of two integrals by removing the parentheses, to express the quantity in square brackets in the second integral in terms of \( d\Phi_3/dV_3 \) and to integrate by parts. It follows

\[ E_3 = \sqrt{2} \left( V_3 - V_s - \theta_3 V_3^{2/3} \left( \frac{1}{V_5^{3/2}} - \frac{1}{V_5^{5/3}} \right) \right) \]

\[ -(1 - \gamma) \left( \frac{1}{V_5} - \frac{e^{\Phi_3}}{V_5} \right)^{1/2}. \] (46)

Substituting this expression into Eq. (43) divided by \( E_3 \) and integrating, one can find a relationship describing the function \( V_3(\eta_3) \).

Now asymptotic analysis is complete. It can be checked that intermediate asymptotic expansion (27) (which includes the two-terms expansions of the ion velocity and temperature and the one-term expansions of \( E \) and \( \Phi \)) satisfies conditions of asymptotic expansions (22) and (36).

VI. RESULTS AND DISCUSSION

Analysis of the preceding section describes three asymptotic zones: the quasineutral Knudsen layer of a thickness \( O(\lambda) \), the collisionless space-charge sheath of a thickness \( O(h) \), and the intermediate transition layer of a thickness \( O(\lambda^{1/5} h^{4/5}) \).

A. Knudsen layer

The Knudsen layer is described by expansion (22). Since \( \Phi \) is small here, the difference between the terms on the right-hand side of the first equation in Eq. (15) is much smaller than each of the terms, hence the plasma is quasineutral in the Knudsen layer. The value of the (dimensional) ion velocity \( v_i \) on the wall side of the layer, defined by the last boundary condition in Eq. (26), equals \( \sqrt{(k/m_i)(T_a + T_e)} \), which is the local ion sound velocity in a quasineutral plasma [here \( T_a = \theta_i(T_a + T_e) \) is the ion temperature on the wall side of the Knudsen layer]. In other words, ions are accelerated in the Knudsen layer from velocities much smaller than the thermal velocity on the plasma side of the layer up to the ion sound velocity on the wall side.

In order to find a distribution of the ion velocity and temperature across the layer, one needs to numerically solve problem (24)--(26). It is convenient to transform the problem to one equation for the function \( \theta_i(V_1) \),

\[ \frac{d\theta_i}{dV_1} = \frac{1}{2V_1^2} \left( \frac{3}{\xi_{ia}} (\theta_1^2 - \gamma^2)(\theta_1 - V_1^2 + 1 - \gamma) + 2 \theta_1 V_1(V_1^2 + 2 \gamma - 1) \right) \] (47)

This equation should be integrated with the boundary condition \( \theta_i(0) = \gamma \) in the direction of increasing \( V_1 \) until the quantity \( \frac{2}{3} \theta_1 - V_1^2 + 1 - \gamma \) vanishes; a value of \( V_1 \) at which the latter happens and a respective value of \( \theta_1 \) will supply \( V_s \) and \( \theta_s \). Note that the singularity which Eq. (47) has at the point \( V_1^2 = 0 \) may be resolved to give

\[ \frac{d\theta_i}{dV_1}(0) = \frac{1 - 2 \gamma}{3}. \] (48)

One can see that the formulated problem is independent of the dimensionless ion mobility \( b \) and the only kinetic coefficient to be specified is \( \xi_{ia} \). Numerical calculations described in this work are performed for two models of ion–atom interaction: the model of rigid spheres and the model of Maxwell molecules (constant frequency of momentum transfer); formulas for \( \xi_{ia} \) for these models follow from results of Ref. 7 and may be written in the form, respectively.

\[ \xi_{ia} = \frac{8M_{ia}^2 \left[ \sqrt{\pi}(1 + 2M_{ia}^2) \operatorname{erf} M_{ia} + 2M_{ia} e^{-M_{ia}^2} \right]}{3 \left[ \sqrt{\pi}(4M_{ia}^4 + 4M_{ia}^2 - 1) \operatorname{erf} M_{ia} + 2M_{ia}(2M_{ia}^2 + 1) e^{-M_{ia}^2} \right]} + \frac{2M_{ia}^2}{3}. \] (49)
and equals 1. 3086 for the model of rigid spheres and 

\( T e \)

the limit of ions relative to atoms.

\( T e \)

equation on the right-hand side of Eq. (3) which describes heating due to conversion of kinetic energy of the ion flow into thermal energy in collisions of ions with atoms) to the first term may be evaluated with the use of Eqs. (7) and equals 

\(- (T_a + T_e) / 2T_a\). It follows that cooling on the plasma side exceeds heating if \( T_e > T_a \), is balanced by heating if \( T_e = T_a \), and is inferior to heating if \( T_e < T_a \). Thus, one should suppose that ions are continuously cooled during their motion to the wall if \( T_e \leq T_a \) and are first heated, then cooled if \( T_e > T_a \), which conforms to the above-discussed numerical results.

One can see that the function \( \theta_1(V_1) \) tends in the limit \( T_e / T_a \to \infty \) not to zero but rather to a finite limit function. In other words, ion temperature in the Knudsen layer in the gas of (asymptotically) cold atoms is finite rather than asymptotically low. This, of course, should have been expected on the basis of the estimates given in Sec. III. On the other hand, the ratio \( T_i / T_e \), while being finite, is rather small (does not exceed approximately 0.08).

One can see that calculation results are virtually independent of the model of ion–atom interaction employed for calculation of \( \xi_{ia} \). The reason may be understood as follows. First, a dependence of \( \xi_{ia} \) on a model of the ion-atom interaction is not very strong. In particular, \( \xi_{ia} \) equals unity at small Mach numbers of the motion of ions relative to atoms regardless of an interaction model; the leading term of the asymptotic expansion of \( \xi_{ia} \) in the opposite limit case of high \( M_{ia} \) (see Ref. 7) is also independent of an interaction model.

Second, an effect produced on the solution by a variation of \( \xi_{ia} \) due to different models of the ion–atom interaction also is not very strong. The reason is that \( \xi_{ia} \) equals 1 regardless of the interaction model on the plasma side of the Knudsen layer where \( M_{ia} = 0 \) (an exception being the case of cold atoms), while on the wall side where the Mach number is the highest the collisional term involving \( \xi_{ia} \) becomes minor.

The distribution of the Mach number of the motion of ions relative to atoms is shown in Fig. 2. One can see that \( M_{ia} \) does not exceed approximately 1.6 at any point of the Knudsen layer in the cases \( T_e / T_a \approx 10 \) and varies from approximately 1.2 on the plasma side of the layer to about 2.8 on the wall side in the limit case of cold atoms. It is of interest to note that the effect of different interaction models on the wall side for \( T_e / T_a = 10 \) is substantially smaller than the effect on the plasma side for \( T_e / T_a \approx 10 \), while the Mach number in the first case is larger than in the second case. A reason is the above-mentioned fact that the collisional term involving \( \xi_{ia} \) is of minor importance on the wall side.

In Fig. 3, values of the dimensionless ion velocity and temperature with which ions arrive at the wall side of the Knudsen layer (and cross the sound barrier and enter the sheath) are shown as functions of \( T_e / T_a \). One can see that \( V_s \) varies in a very narrow range, which is due to the normalization employed. On the contrary, the variation of \( \theta_i \) is quite substantial (about an order of magnitude). Note that numerical results for \( \theta_i \) shown in Fig. 3 may be approximated with an error within 7% in the whole range of \( T_e / T_a \).
by means of the formula
\[ \theta_s = 0.6\left(V_s^2 - 1 + \gamma\right) \]
in which \( V_s^2 \) is set constant and equal to 1.12.

In Fig. 4, the ratio of the ion temperature on the wall side to the electron temperature is shown. In the case \( T_a = T_e \), \( T_i/T_e \) is about 0.8; \( T_i/T_e \) decreases with a decrease of \( T_a/T_e \) and tends in the limit case of cold atoms to a finite value equal to 0.070 for rigid spheres and to 0.077 for Maxwell molecules.

Also shown in Fig. 4 is the ratio of the atom temperature to \( T_e \). One can see that the ion temperature on the wall side of the Knudsen layer is below the atom temperature if \( T_e \cdot 3.5T_a \) and exceeds the atom temperature if \( T_e \) and 3.5\( T_a \).

Taking into account that both temperatures coincide on the plasma side, one can say that ions are cooled on having crossed the Knudsen layer in the first case and are heated in the second case.

The ion velocity and temperature on the wall side of the Knudsen layer for the model of rigid spheres exceed respective values for Maxwell molecules at low \( T_e/T_a \) (although the difference in the temperature values is hardly visible on the graph) and are below those values at high \( T_e/T_a \). This result may be understood as follows: coefficient \( \xi_{ia} \) governing rate of energy exchange between ions and atoms due to a difference of their temperatures is higher for the model of rigid spheres and it is natural to expect that the ion temperature for this model is closer to the atom temperature than the ion temperature for Maxwell molecules; the difference in \( \theta_s \) for the two models results in a similar difference in \( V_s \).

Note, however, that the differences are rather small, which is similar to the results shown in Figs. 1 and 2.

It is of interest to calculate the density of the energy flux brought to an absorbing wall by neutral particles diffusing to it. Using the value of \( \theta_s \) in the limit \( T_e/T_a = 0 \) which equals 0.722 for both models of rigid spheres and Maxwell molecules, one finds by means of the second equation in Eq. (20) that this density equals \( 2.41f/kT_a \). Note that this value is rather close to the density of the energy flux in a diffusion-dominated gas far away from the wall, which is determined by the enthalpy transport and equals \( 2.5f/kT_a \).

B. Transition region

Expansion (27) describes the intermediate transition region which separates the Knudsen layer from the sheath. The ion velocity in the transition region is close to the local ion sound velocity (Bohm criterion).

The space charge and collisions are minor effects here, which can be seen from the fact that \( \Phi \) is small in this region while Eqs. (28) and (29) do not include the collisional terms. Since, however, major effects cancel [i.e., Eqs. (28)–(30) are dependent], both the space charge and collisions produce a finite effect on the solution [cf. Eq. (31); more precisely, they produce a finite effect on \( E \) and \( \Phi \), while \( V \) and \( \theta \) are to a first approximation constant across the transition region].
C. Space-charge sheath

The space-charge sheath is described by expansion (36). The friction force and the change of energy of ions due to their collisions with atoms are minor effects here. Ions are accelerated to supersonic velocities inside the sheath. The ion temperature in the sheath decreases adiabatically, cf. Eq. (42).

Note that the variation of the potential in the transition layer is much smaller than characteristic values of the potential inside the sheath. This allows one to introduce the voltage drop across the sheath, \( U \), defined as the difference between the value of the potential in the transition layer, equal (to a first approximation) to \( (kT_e/e) \ln (J_i/n_iu_iV_i) \), and the value at the surface. It is natural to use \( U \) as a control parameter instead of \( \chi \) while considering the sheath solution. It is natural also to express the sheath solution in terms of the ion temperature at the sheath edge, \( T_i \), instead of \( T_e \). Introducing the new dimensionless ion velocity at the wall by the formula \( W = v_{in}/\sqrt{kT_is/m_i} \), one can write the equation relating \( W \) to \( U \) at first approximation in the form

\[
W^2 - \left( \frac{5}{3} + \beta \right) \frac{1}{2\beta} \left[ 1 - \left( \frac{5}{3} + \beta \right) \frac{1}{W} \right]^{2/3} = \frac{eU}{kT_e},
\]

where \( \beta = T_e/T_is \).

The electric field and the ion energy flux density at the surface to a first approximation may be expressed by means of the formulas

\[
\frac{e_0}{2J_i\sqrt{m_i kT_e}} \sqrt{1 + \frac{5}{3\beta} \left( \frac{d\phi}{dy} \right)_w^2}
= \frac{\sqrt{\frac{5}{3} + \beta}}{\beta} \left[ W^2 - 2 - \frac{8}{3\beta} + \left( 1 + \frac{5}{3} + \beta \right) \frac{1}{W} \right]^{5/3}
+ \exp \left( -\frac{eU}{kT_e} \right),
\]

\[
q = J_i \left( \frac{10}{3} kT_is + \frac{kT_e}{2} + eU \right).
\]

VII. COMPARISON OF FLUID AND FREE-FALL SOLUTIONS FOR THE SPACE-CHARGE SHEATH

The ion motion is described by means of the fluid model in the present work. As it is noted in Ref. 3 in connection with the investigation of a low-pressure glow discharge column, what gives the model its usefulness is the fact that the results of the fluid model give a smooth transition from a good agreement with the free-fall model to identically the collision-dominated model; thus the fluid model spans the whole range of conditions from collision-free to collision-dominated plasma column. As to the problem of a transition from a collision-dominated plasma to a negative wall considered in the present work, one can hope that the fluid model spans the range of conditions from collision-free to collision-dominated space-charge sheath. In this connection, it is essential to check whether the above solution for the sheath, obtained on the basis of the fluid model, agrees with a free-fall solution.

We shall consider an approximate free-fall solution obtained in Ref. 13. The respective formulas of Ref. 13 modified in such a way that the mean ion velocity at the sheath edge be equal to \( \sqrt{(k/m_i)/(2T_is + T_e)} \) (which amounts to setting the quantity \( u_i \) in analysis of Ref. 13 equal to \( \sqrt{5kT_is/3m_i} \)) may be written as

\[
\frac{e_0}{2J_i\sqrt{m_i kT_e}} \sqrt{1 + \frac{5}{3\beta} \left( \frac{d\phi}{dy} \right)_w^2}
= \frac{\sqrt{\frac{5}{3} + \beta}}{\beta} \left[ 2(W^2 + W_-W_+ + W^2) \right]
- 2 - \frac{20}{9\beta} + \exp \left( -\frac{eU}{kT_e} \right),
\]

\[
q = J_i \left( \frac{8}{3} kT_is + \frac{kT_e}{2} + eU \right).
\]

Equations (54), (55) and (56), (57) become to a first approximation identical in the limit case when \( T_is \) is much smaller than \( T_e \) and/or than \( eU/k \). This, of course, should have been expected: the ions constitute in this case a monoenergetic beam which is adequately described both by Eq. (2) (in which the first and last terms on the right-hand side are negligible and which thus assumes the form of the equation of motion of an ion) and by the kinetic treatment of Ref. 13 (in which the effect of the used in Ref. 13 approximation of the distribution of ions entering the sheath by a rectangular function becomes negligible).

A quantitative comparison between formulas (54), (55) and (56), (57) is shown in Fig. 5. The results are given for the case when the potential of the surface is below the floating potential,

\[
\frac{eU}{kT_e} \geq \frac{1}{2} \ln \frac{m_iT_e}{2\pi m_e \left( \frac{5}{3} T_is + T_e \right)}
\]

(59)

(here \( m_e \) is the electron mass). Note that the inequality (59) has been obtained with the use of the conventional expression for density of the electron flux to the surface (see, e.g., Ref. 14),

\[
J_e = \frac{n_is}{4} \left( \frac{8kT_e}{\pi m_e} \right)^{1/2} \exp \left( -\frac{eU}{kT_e} \right),
\]

(60)

where \( n_is = J_i/u_iV_i \) is the charge particle density at the edge of the space-charge sheath.

One can see that the difference between the fluid and kinetic values decreases with an increase of \( T_e/T_is \) and/or \( eU/kT_e \) in agreement to the above considerations. On the other hand, this difference is relatively small even in the case...
of the ratio $T_e/T_i$, equal to 1.25 (which corresponds to $T_e = T_a$ and is the lowest value likely to be encountered in practical calculations) and of the floating potential; it amounts to approximately 2% for the electric field and to approximately 8% for the ion energy flux.

The dash-and-dotted line in Fig. 5 represents the ratio of the value of ion energy flux density at the surface calculated in the fluid approximation under the assumption of constant (and equal to $T_e$) ion temperature inside the sheath, to the kinetic value. In other words, this line is a counterpart of the dashed line marked $T_e/T_i = 1$ calculated without regard of variability of the ion temperature in the sheath. One can see that the account of variability of $T_i$ introduced in the present work results in a substantially smaller difference between the fluid and free-fall solutions.

VIII. CONCLUSIONS

The problem of plasma–wall transition with account of variability of the ion temperature has been formulated. An asymptotic solution was obtained for the limit case of small ratio of the Debye length to the characteristic mean free path for collisions ion-neutral.

Asymptotic structure of the solution is similar to that found in previous works under the assumption of asymptotically cold ions (see, e.g., Ref. 1, and references therein) and includes the quasineutral Knudsen layer, the transition layer, and the collision-free space-charge sheath. A solution in the Knudsen layer is described by the initial-value problem (47), (48), which has been solved numerically. It was found that a dependence of the solution on a model of the ion–atom interaction is rather weak. If $T_e > T_a$, the distribution of the ion temperature in the Knudsen layer is nonmonotonic. The reason is that heating of ions due to conversion of kinetic energy of the ion flow into thermal energy in collisions of ions with atoms prevails on the plasma side of the layer, while on the wall side a dominating effect is cooling due to work of the pressure force. For the particular case when the ratio $T_e/T_a$ is asymptotically small, it was found that $T_i/T_e$, although finite rather than asymptotically small, is still small: its maximum value does not exceed approximately 0.08.

The ion velocity and temperature at the edge of the space-charge sheath may be either taken from numerical calculations for the Knudsen layer, or estimated by means of the correlation formulas

$$v_{is} = 1.08\sqrt{k(T_a + T_e)/m_i}, \quad T_{is} = 0.672 T_a + 0.072 T_e, \quad (61)$$

which describe numerical results obtained in this work with the accuracy of 2% and 7%, respectively. After $v_{is}$ and $T_{is}$ have been found, parameters of the space-charge sheath can be determined. In particular, one can find the dimensionless ion velocity, electric field and ion energy flux density at the surface by means of Eqs. (53), (54), and (55).

ACKNOWLEDGMENTS

The work has been supported by the program PRAXIS XXI, FEDER, and CITMA.