Collision-dominated to collisionless electron-free space-charge sheath in a plasma with variable ion temperature

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A theory of the near-cathode space-charge sheath in the case when the near-cathode voltage is high enough and the presence of electrons in the sheath is unessential is developed on the basis of a fluid description of the ion motion with account of ion-atom collisions and of variable ion temperature. The model spans the range of conditions from a collision-free to collision-dominated space-charge sheath. Detailed analytical and numerical results are presented for two models of ion-atom interaction, the model of rigid spheres (constant mean free path) and the model of Maxwell molecules (constant frequency of momentum transfer). It is found, in particular, that the assumption of cold ions provides a good accuracy in the problem considered. An analytical solution has been obtained under this assumption for the model of Maxwell molecules. © 2000 American Institute of Physics. [S1070-664X(00)03611-9]

I. INTRODUCTION

The problem of a theoretical description of the near-cathode space-charge sheath continues to receive a great deal of attention in the literature, in particular, in connection with the calculation of near-cathode layers in high-pressure arc discharges (e.g., Refs. 1–3). The most important parameter which governs the physics of the sheath is the ratio of the mean free path for ion-atom collisions to the sheath thickness. At present, the theory is well developed for the limit cases when this ratio is large (the case of a collisionless sheath) or small (the case of a collision-dominated sheath). Fundamentals of the theory for the first and second cases were established in Refs. 4–6 and 7, 8, respectively; reviews of subsequent works can be found in, respectively, Refs. 9, 10 and Refs. 11, 12.

Both abovementioned limit cases are of practical interest; as far as arc discharges are concerned, the case of a collision-free sheath is typical for low plasma pressures and/or a spot mode of current transfer to the cathode, while the case of a collision-dominated sheath is typical for high pressures and a diffuse mode. It is important, of course, to develop a theory also for the intermediate case, when the mean free path for ion-atom collisions is comparable to the sheath thickness. Furthermore, it is desirable to develop a unified theory applicable to arbitrary values of the ratio mean free path/sheath thickness; this is especially important in connection with development of numerical codes describing the whole system arc electrodes.

A necessity of development of a model valid in the case when the mean free path for ion-atom collisions is comparable to the sheath thickness has been realized long ago. A number of (unsuccessful) attempts have been made to generalize for the case of a collisional sheath the Bohm criterion (e.g., Refs. 13–16). An approximate approach valid in the range of values of the ratio of the ion-atom mean free path to the Debye length exceeding unity was suggested in Ref. 17. A numerical solution for a collisionless to collisional active plasma for the case of cold ions has been obtained in Ref. 18.

The above-cited works 13–18 are based on the fluid description of ion motion in the sheath. Another assumption used in Refs. 13–17 was that the electron density in the sheath obeys the Boltzmann distribution. The latter assumption implies that the sheath voltage is large enough to repel most of the plasma electrons and is quite natural as long as sheaths at floating walls or cathodes are treated. A natural further step is to explicitly introduce a large parameter related to the sheath voltage and to employ an appropriate asymptotic technique. In other words, one can apply an asymptotic approach based on treating the sheath voltage as a large parameter to the problem of a theoretical description of a near-cathode space-charge sheath without using any assumptions concerning the ratio of the ion-atom mean free path to the sheath thickness. As far as high-pressure arc discharges are concerned, such an approach seems natural also in connection with direct measurements of the near-cathode voltage drop reported recently in Ref. 19, which revealed that the voltage drop is such discharges may be up to 40 V, i.e., much higher than most workers previously believed on the basis of indirect considerations and estimates.

In Refs. 7, 20–22, an asymptotic approach based on treating the sheath voltage as a large parameter has been applied to the case of a collision-dominated sheath. In the case of a collision-free sheath formed by cold ions, it was shown (Ref. 23) that such an approach allows one to develop a simple model with an exponentially small error which provides accuracy of several per cent for sheath voltages exceeding $3 kT_e/e$ (here $T_e$ is the electron temperature). The latter example indicates that an asymptotic approach based on treating the sheath voltage as a large parameter can provide a good accuracy not only for near-cathode sheaths, but in some cases also for surface potentials about, or even above, the floating potential.

A key element within such an approach is a theory of a collisionless to collision-dominated electron-free space-
charge sheath. Formerly, such a theory was developed for the case of cold ions interacting with atoms as rigid spheres; see Ref. 24. In the present work, the theory is developed with account of finite (and variable) ion temperature and for a more generic potential of ion-atom interaction. In Sec. II, a system of governing equations is given. A model of the space-charge sheath is stated in Sec. III. In Sec. IV, a solution is studied for small and large values of the sheath voltage. An analytical solution is presented for the case of cold ions for the model of constant frequency of momentum transfer for ion-atom collisions. Results of numerical calculations are given and discussed in Sec. V. Concluding remarks are given in Sec. VI.

II. GOVERNING EQUATIONS

Consider an atomic plasma with the ionization degree much less than unity at rest near a negative electrode (cathode). The space region under consideration includes a thin layer adjacent to the cathode surface in which the space charge effects and ion inertia are localized; outside the layer the plasma is quasineutral and the ion motion is collision-controlled. Equations describing motion of the ion fluid read

\[ n_i v_i = J_i, \]

\[ m_i n_i v_i \frac{d v_i}{d y} = -\frac{d(n_i k T_i)}{d y} - e n_i \frac{d \phi}{d y} + e n_i \frac{d \phi}{d y}, \tag{2} \]

\[ \frac{d}{d y} \left( \frac{3 k T_i}{2} \right) = -k T_i \frac{d v_i}{d y} + \frac{3 \xi_{ia} e k \mu_i (m_i + m_a) (T_i - T_a)}{\mu_i (m_i + m_a) (T_i - T_a)} - \frac{e m_i T_i v_i^2}{\mu_i (m_i + m_a) (T_i - T_a)}. \tag{3} \]

Here the y-axis is directed from the cathode into the plasma, \( n_i \) is the number density of ions, \( v_i \) is the mean velocity of the motion of ions in the direction to the surface, \( J_i \) is the density of the ion flux to the surface (a given constant), \( T_i \) is the ion temperature, \( \phi \) is the electrostatic potential, \( \mu_i \) is the ion mobility, \( \xi_{ia} \) is a dimensionless coefficient, \( T_a \) is the temperature of atoms (a given constant), and \( m_i \) and \( m_a \) are particle masses of ions and atoms. Note that equations of conservation of number and momentum of ions Eqs. (1) and (2) are written in a usual form. The second and third terms on the right-hand side of Eq. (3) describe, respectively, the exchange of thermal energy between ions and atoms and heating of ions due to conversion of kinetic energy of their directed motion into thermal energy in elastic collisions with atoms; these terms have been deduced in the first approximation in the two-temperature displaced-distribution theory for ion mobility (Ref. 25, p. 203). Expressions for \( \mu_i \) and \( \xi_{ia} \) in terms of integrals of the weighted cross section for momentum transfer in ion-atom collisions, deduced in the same approximation, can be found from Ref. 26.

The electron density obeys the Boltzmann distribution

\[ n_e = n_0 \exp \left( \frac{e \phi}{k T_e} \right), \tag{4} \]

where \( T_e \) the electron temperature is assumed to be constant in the near-cathode region considered and \( n_0 \) is a constant which should be determined with the use of boundary conditions (see Ref. 27).

The system is closed by the Poisson equation

\[ \varepsilon_0 \frac{d^2 \phi}{d y^2} = -e (n_i - n_e). \tag{5} \]

Assuming that the neutral gas pressure and temperature are given, one can consider (in the framework of the first approximation in the two-temperature displaced-distribution theory) ion mobility \( \mu_i \) and coefficient \( \xi_{ia} \) as known functions of \( T_i \) and \( v_i \). It is more convenient, however, to use as arguments \( T_i \) and the Mach number \( M_{ia} = (\sqrt{m_i m_a}/2k(m_i T_i + m_a T_a)) v_i \) of the motion of the ions relative to the atoms: \( \mu_i = \mu_i(T_i, M_{ia}) \), \( \xi_{ia} = \xi_{ia}(T_i, M_{ia}) \).

In the particular case when the ions interact with the atoms as rigid spheres (the model of constant mean free path), the ion mobility may be written as

\[ \mu_i = \frac{c_1 \sqrt{m_a}}{\xi_{ia} \sqrt{T_i T_a + m_a T_a}}, \tag{6} \]

Function \( \xi_{ia} = \xi_{ia}(M_{ia}) \) introduced here may be found from Eq. (38) of Ref. 26 and constant \( c_1 \) is

\[ c_1 = \frac{3 \sqrt{2 \pi} m_i + m_a}{16} \frac{m_i + m_a}{\sqrt{km_i}} \tag{7} \]

where \( \lambda_i \) is the mean free path of ions, related to \( n_a \) the number density of atoms and \( Q_{ia}^{(1)} \) the (constant) cross section for momentum transfer by the formula \( \lambda_i = 1/n_a Q_{ia}^{(1)} \).

Note that \( \xi_{ia} (0) = 1 \); asymptotic behavior of function \( \xi_{ia}(M_{ia}) \) at \( M_{ia} \to \infty \) is \( \xi_{ia} \sim 3 \sqrt{\pi} M_{ia}/8 \).

In the particular case when the ions interact with the atoms as Maxwell molecules (the model of constant frequency of momentum transfer), the mobility is constant and may be expressed as

\[ \mu_i = \left( \frac{1}{m_i} + \frac{1}{m_a} \right) \frac{e}{v_i}, \tag{8} \]

where \( v_i = n_a g Q_{ia}^{(1)}(g) \) is the frequency of momentum transfer [here \( Q_{ia}^{(1)}(g) \) designates the velocity-dependent cross section for momentum transfer, which is, in the framework of the model of Maxwell molecules, inversely proportional to the relative velocity \( g \) of the particles].

In a general case, \( \xi_{ia}(T_i, 0) = 1 \) and \( \xi_{ia} \approx 2 M_{ia}^2/3 \) as \( M_{ia} \to \infty \). For the models of rigid spheres and Maxwell molecules, \( \xi_{ia} \) is a function of \( M_{ia} \) only. This function may be found from Eq. (39) of Ref. 26 for rigid spheres and is \( \xi_{ia} = 1 + 2 M_{ia}^2/3 \) for Maxwell molecules.

For the following analysis, we need to consider a particular form of Eqs. (2) and (3) in a spatially uniform case with electric field. The differential terms vanish and these equations may be written as

\[ v_i = \frac{\mu_i d \phi}{d y}, \tag{9} \]

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The first equation is the conventional hydrodynamics expression for the drift velocity; the second equation describes a collision-dominated mobility-controlled energy lost in elastic collisions with the atoms (note that the second factor on the right-hand side can be interpreted as an effective frequency of elastic ion-atom collisions). These equations describe a collision-dominated mobility-controlled ion motion in the framework of the first approximation in the two-temperature displaced-distribution theory.

We shall need also a solution to Eq. (10) for a particular case of cold atoms, $T_a \ll T_1$. In this case, the equation assumes the form

$$3 \xi_{ia} = \frac{m_i + m_a}{m_i} M_{ia}^2. \quad (11)$$

If $\xi_{ia}$ is a function of $M_{ia}$ only, then Eq. (11) is closed and may be solved for $M_{ia}$. The respective root will be denoted by $c_2$. A solution for Maxwell molecules is $c_2 = \sqrt{3m_i/2m_a}$. For the model of rigid spheres, Eq. (11) should be solved numerically; e.g., $c_2 \approx 1.3086$ if $m_i = m_a$. The drift velocity given by Eq. (9) for the model of rigid spheres in the case of cold atoms is

$$v_i = \frac{c_3 (3\pi)^{1/4}}{4} \left( \frac{m_i + m_a}{m_i} \right) \left( \frac{eJ_i}{m_i} \right)^{1/2} \frac{d\phi}{dy}, \quad (12)$$

where

$$c_3 = \left( \frac{m_i + m_a}{m_i} \right)^{1/4} \left( \frac{c_2}{\xi_{ia}(c_2)} \right)^{1/2}. \quad (13)$$

Note that $c_3 \approx 1.1898$ in the case $m_i = m_a$.

It is of interest to note that Eqs. (9) and (10), describing balance of momentum and energy obtained by the ion fluid from the electric field and lost in elastic collisions with the atoms, are essentially the same as on which the momentum-transfer theory for ion mobility (e.g., Ref. 25, p. 145) is based. Due to different averaging, results given by the momentum-transfer theory differ from those which follow from Eqs. (9) and (10), however the difference is generally not too large. For example, Eq. (10) for the model of Maxwell molecules may be written in the form

$$\frac{3}{2} kT_i = \frac{3}{2} kT_a + m_i v_i^2, \quad (14)$$

which coincides with a formula for the average random energy of the ions derived by means of the momentum-transfer theory [the Wannier formula; cf. Eq. (5-2-20) of Ref. 25 in which the difference $\frac{1}{2}m_i v_i^2 - \frac{1}{2}m_i v_i^2$ is designated by $3kT_i/2$]. A momentum-transfer formula for the drift velocity in the gas of cold atoms for the model of rigid spheres can be derived from Eq. (5-2-23) of Ref. 25 and is similar to Eq. (12), however without factor $c_3 (3\pi)^{1/4}$. Since this factor is not very different from unity (if, for example, $m_i = m_a$, then it equals 0.97), the two formulas are rather close.

### III. A MODEL OF THE SHEATH

The term “sheath” refers conventionally to a layer adjacent to a solid surface in which the space charge is localized. If the surface is under a large enough negative potential, the electron density in the bulk of the sheath is much smaller than the ion density, hence the electron space charge may be neglected in comparison with the ion charge. The electric field in the bulk of the sheath is large and the kinetic energy of directed motion of the ions in most cases exceeds substantially the thermal energy of the atoms. Therefore, the temperature of the atoms will be set equal to zero. For simplicity, it is assumed also that $m_i = m_a$. Equations (1) and (2) do not change the appearance; Eqs. (3) and (5) may be written in the form

$$v_i \frac{d}{dy} \frac{3kT_i}{2} = -kT_i \frac{d}{dy} v_i + \frac{3 \xi_{ia} e kT_i}{2 \mu_i m_i} - \frac{e v_i^2}{\mu_i}, \quad (15)$$

$$\xi_{ia} \frac{d^2 \phi}{dy^2} = -\frac{e J_i}{v_i}. \quad (16)$$

The expression for the Mach number of the ion motion and Eq. (6) assume the form $M_{ia} = v_i / \sqrt{2kT_i}$, $\mu_i = c_1 \xi_{ia} \sqrt{T_i}$.

A boundary condition at the cathode surface is obtained by specifying the local potential,

$$\phi = -U, \quad (17)$$

where $U$ is the sheath voltage (a given quantity).

Assume that the sheath has a more or less distinct boundary on the plasma side and one can speak of the sheath edge. It is convenient to choose zero of the $y$-axis at this edge. Then the boundary condition Eq. (17) applies at $y = -y_D$, where $y_D$ is an unknown parameter which has the meaning of the sheath thickness and should be determined as a part of the solution.

One can assume that values of the ion velocity, of the electric field, and of the ion temperature in the vicinity of the edge are much smaller than respective values in the bulk of the sheath and that the sheath gives a major contribution to the total voltage drop in the near-cathode region considered. Then the boundary conditions for Eqs. (1), (2), (15), and (16) at the sheath edge read

$$y = 0: \quad v_i = 0, \quad \frac{d\phi}{dy} = 0, \quad T_i = 0, \quad \phi = 0. \quad (18)$$

Boundary conditions similar to Eq. (18) have been used by many previous authors. In the particular case of a collisionless sheath formed by cold ions, boundary conditions used in the well-known Child–Langmuir model are those of zero electric field and zero ion velocity at the sheath edge; see Refs. 4, 5. In the particular case of a collision-dominated mobility-controlled sheath, the boundary condition of zero electric field has been used; see, e.g., Refs. 28, 29. This boundary condition was employed also in Ref. 30, in which a highly collisional sheath was treated with the use of an approximate kinetic approach. In Ref. 24, boundary conditions of zero of kinetic energy of directed motion of the ions and of its derivative were imposed in the case of a collision-
free to collision-dominated sheath formed by cold ions interacting with the atoms as rigid spheres; these boundary conditions are equivalent to zero electric field and zero ion velocity. On the other hand, some authors have employed more complex boundary conditions; see, e.g., Refs. 23, 31–33 for the case of a collisionless sheath formed by cold ions and Refs. 34, 35 for a collision-dominated mobility-controlled sheath.

From the point of view of an asymptotic approach based on considering $eU/kT_e$ as a large parameter, boundary conditions Eq. (18) correspond to the first approximation. In fact, if $eU/kT_e$ takes values of the order of $10^3$ or higher, typical for sheaths on cold cathodes, an exact form of boundary conditions being imposed at the sheath edge is not of crucial importance. For example, one can set the ion velocity at the edge of a collisionless sheath formed by cold ions equal to zero or to the Bohm velocity, however, the solution will not change dramatically. On the other hand, an appropriate choice of boundary conditions may considerably improve the accuracy in cases when $eU/kT_e$ is not very high. However, the question of constructing enhanced-accuracy boundary conditions is left beyond the scope of this work and simple boundary conditions Eq. (18) will be used.

Now the statement of the model is complete. It is convenient to exclude $n_i$ by means of Eq. (1) and to introduce dimensionless variables

$$
\eta = \frac{eV}{m_i\mu_i(0)v_i(0)} , \quad V = \frac{v_i}{v_i(0)} , \quad E = \frac{\mu_i(0)}{v_i(0)} \frac{d\Phi}{dy},
$$

$$
\Phi = \frac{e\Phi}{m_i\mu_i(0)^2} , \quad \theta = \frac{kT_i}{m_i\mu_i(0)^2},
$$

where the characteristic ion velocity is defined as $v_i(0) = m_i\mu_i(0)^2 J_i/\varepsilon_0$ and $\mu_i(0)$ is a characteristic ion mobility. The problem assumes the form

$$
dV = -V \frac{d(\theta V)}{d\eta} - E + \frac{V}{b} , \quad dE = -1 , \quad \frac{d\Phi}{d\eta} = E, \tag{20}
$$

$$
\frac{3}{2} \frac{dV}{d\eta} = -\theta \frac{dV}{d\eta} + \frac{3\xi_i \theta}{2b} - \frac{V^2}{b}, \tag{21}
$$

$$
V(0) = 0, \quad E(0) = 0, \quad \theta(0) = 0, \tag{22}
$$

where

$$
b = \frac{\mu_i}{\mu_i(0)}, \quad \eta_D = \frac{e_0 e V_D}{m_i\mu_i(0)^2 J_i}, \quad \tilde{U} = \frac{e_0^2 eU}{m_i\mu_i(0)^2 J_i}. \tag{23}
$$

The expression for the Mach number of the ion motion in the dimensionless variables becomes $M_{ia} = V/\sqrt{2}\theta$. Assuming for definiteness that the characteristic ion mobility is defined by equation $\mu_i(0) = \mu_i(m_i\mu_i(0)^2/k,0)$, one finds

$$
\mu_i(0) = \left( \frac{e_0^2 e k \xi_i}{m_i^2 J_i} \right)^{1/6} , \quad b = \frac{1}{\sqrt{\theta^2 \xi_i}} ; \quad \mu_i(0) = \mu_i , \quad b = 1 \tag{24}
$$

for the models of rigid spheres and of Maxwell molecules, respectively.

It is convenient for the following analysis to transform the problem to new independent variable $E$

$$
\frac{dV}{dE} - \frac{d(\theta V)}{dE} = E - \frac{V}{b} , \quad \Phi = - \int_0^E V dE , \tag{25}
$$

$$
\eta = - \int_0^E V dE , \tag{26}
$$

$$
\frac{3}{2} \frac{dV}{dE} + \frac{\theta V}{E} = - \frac{3\xi_i \theta}{2b} + \frac{V^2}{b} , \tag{27}
$$

for any model of the ion-atom interaction of the class considered. Using these formulas, one finds that the asymptotic behavior of functions $V(\eta), E(\eta)$, and $\Phi(\eta)$ at $\eta \to 0 - 0$ to a first approximation is

$$
V = \frac{2}{3} \eta^{2/3} \frac{2}{3} \eta^{2/3}, \quad E(\eta) = -\frac{6}{3} \eta^{1/3}, \quad \Phi(\eta) = -\frac{3^{4/3}}{2^{3/3} \eta^{4/3}}. \tag{28}
$$

It follows from the above that $M_{ia} \to \infty$ as $E \to 0$. Using the asymptotic behavior of coefficient $\xi_i$ at $M_{ia} \to \infty$, one finds that $\xi_i = V^2/3\theta$ to a first approximation. A solution to Eq. (26) reads.

IV. ANALYTICAL TREATMENT

We start with finding an asymptotic solution of problem Eqs. (20)–(22) at $\eta \to 0 - 0$, i.e., in the vicinity of the sheath edge. We assume that the second term on the right-hand side of Eq. (26) is at small $E$ of the same order of magnitude as the derivatives on the left-hand side. (As is usual in asymptotic analysis, the validity of this assumption will be confirmed by the self-consistency of the resulting asymptotic solution and by its agreement with numerical results.) One finds $\theta/V^2 = O(E/b)$. The second term on the left-hand side of the first equation in Eq. (25) is comparable to the second term on the right-hand side and has the order of $E/b$ relative to the first term on the left-hand side.

One needs to know the order of magnitude of $1/b$ at small $E$ in order to make use of this estimate. Consider first the model of rigid spheres. If $M_{ia}$ is finite at small $E$, $b$ is of the order of $\theta^{-1/2}$. If $M_{ia} \to \infty$ as $E \to 0$, it follows from the asymptotic behavior of coefficient $\xi_i$ at $M_{ia} \to \infty$ that $b = 8(\sqrt{2}/3)\sqrt{\pi} V$. In both cases, $1/b$ tends to zero as $E \to 0$. In the framework of the model of Maxwell molecules, $1/b$ is finite. Restricting the consideration to the class of models intermediate between rigid spheres and Maxwell molecules, one can expect that $1/b = O(1)$ at small $E$ for any model of the class considered.

It follows that the second term on the left-hand side and the second term on the right-hand side of the first equation in Eq. (25) may be dropped at small $E$. One finds to a first approximation

$$
V = \frac{E^2}{2}, \quad \Phi = -\frac{E^2}{8} , \quad \eta = -\frac{E^3}{6} \tag{29}
$$

for any model of the ion-atom interaction of the class considered. Using these formulas, one finds that the asymptotic behavior of functions $V(\eta), E(\eta)$, and $\Phi(\eta)$ at $\eta \to 0 - 0$ to a first approximation is

$$
V = \frac{2^{2/3}}{3^{1/3}} \eta^{2/3}, \quad E(\eta) = (-6\eta)^{1/3}, \quad \Phi(\eta) = -\frac{3^{4/3}}{2^{3/3} \eta^{4/3}}. \tag{29}
$$

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Evaluating the integral for the models of rigid spheres and Maxwell molecules, one finds to a first approximation the following asymptotic behavior of function $\theta(E)$ at small $E$, respectively,

$$\theta = \frac{3}{1600} \sqrt{\frac{\pi}{2}} E^7, \quad \theta = \frac{E^5}{76}.$$  \hspace{1cm} (31)

Asymptotic behavior of function $\theta(\eta)$ at $\eta \rightarrow 0$ to a first approximation is

$$\theta = \frac{2^{5/3} 3^{10/3} \sqrt{\pi}}{800} (-\eta)^{7/3}, \quad \theta = \frac{3^{5/3}}{2^{1/3} 19} (-\eta)^{5/3}$$  \hspace{1cm} (32)

for the models of rigid spheres and Maxwell molecules, respectively.

Proceed to finding an asymptotic solution at large $\eta$. Supposing that the differential terms of the first equation in Eq. (25) and of Eq. (26) are much smaller than the terms on the right-hand sides, one finds that $M_{ia}$ tends at $E \rightarrow \infty$ to a finite limit equal to $c_2$. One finds to a first approximation

$$V = c_3 \sqrt{E}, \quad \theta = \frac{c^2_3}{2 c^2_2} E, \quad \Phi = -\frac{2 c^2_3}{5} E^{5/2},$$  \hspace{1cm} (33)

for the model of rigid spheres and

$$V = E, \quad \theta = \frac{E^2}{3}, \quad \Phi = -\frac{E^3}{3}, \quad \eta = -\frac{E^2}{2}$$  \hspace{1cm} (34)

for the model of Maxwell molecules. Asymptotic behavior of functions $V(\eta), E(\eta), \theta(\eta),$ and $\Phi(\eta)$ at $\eta \rightarrow -\infty$ is to a first approximation

$$V = \left( -\frac{3 c^2_3}{2} \eta \right)^{1/3}, \quad E = \left( \frac{3}{2 c^2_2} \eta \right)^{2/3},$$

$$\theta = \left( \frac{3 c^2_3}{2} \eta \right)^{2/3}, \quad \Phi = -\left( \frac{3 c^2_3}{2} \eta \right)^{5/3}$$  \hspace{1cm} (35)

for the model of rigid spheres and

$$V = (-2 \eta)^{1/2}, \quad E = (-2 \eta)^{1/2},$$

$$\theta = -\frac{2 c^2_3}{3}, \quad \Phi = -(-2 \eta)^{3/2}$$  \hspace{1cm} (36)

for the model of Maxwell molecules.

If $\tilde{U}$ is small, $\eta_D$ is small as well and asymptotic formulas Eqs. (29) and (32) apply to a first approximation, throughout the sheath. In other words, an approximate solution of problem Eqs. (20)–(22) at small $\tilde{U}$ is given by Eqs. (29) and (32). The sheath thickness may be found to be $\eta_D = (32 \tilde{U}^3/81)^{1/4}$. In dimensional variables, the solution for the ion velocity, the electric field, the potential, and the sheath thickness reads

$$v_i = \left( \frac{9 e^2 J_i^4 \gamma^2}{2 e_0 m_i} \right)^{1/3}, \quad \Phi = -\left( \frac{6 e m_i J_i^2 \gamma}{e_0} \right)^{1/3}.$$  \hspace{1cm} (37)

Solution Eq. (37) satisfies Eqs. (2) and (16) without the first and last terms on the right-hand side of Eq. (2). In other words, both ion pressure gradient and ion-atom collisions produce a minor effect on the ion motion at small $\tilde{U}$ and the ions can be considered to a first approximation as a monoenergetic beam. As it should have been expected, these formulas coincide with the Child–Langmuir solution (Refs. 4, 5) describing a free-fall sheath formed by ions entering the sheath with a zero velocity.

The solution for the ion temperature at small $\tilde{U}$ in dimensional variables reads for the models of rigid spheres and Maxwell molecules, respectively,

$$kT_i = \frac{2^{5/3} 3^{10/3} \sqrt{\pi}}{800} \left( \frac{e^{14} J_i^{14} f_i}{e_0^4 kg_m c_6^6} \right)^{1/6},$$

$$kT_i = \frac{3^{5/3}}{2^{1/3} 19} \left( \frac{e^5 J_i^5 \gamma^5}{e_0 m_i c_6} \right)^{1/3}$$  \hspace{1cm} (38)

In contrast to Eq. (37), the formulas in Eq. (38) involve quantities depending on a cross section of ion-atom collisions ($c_1$ or $\mu_i$, respectively). Note that all the terms of Eq. (21) are at small $\eta$ of the same order, which means that all the terms of Eq. (15) are of the same order at small $\tilde{U}$. In other words, at small $\tilde{U}$ collisions of the ions with the atoms, while producing a negligible effect on the ion motion, affect considerably the ion temperature.

If $\tilde{U}$ is large, $\eta_D$ is large as well and asymptotic formulas Eqs. (35) or, respectively, Eq. (36) apply in the bulk of the sheath. In other words, an approximate solution of problem Eqs. (20)–(22) at large $\tilde{U}$ is given by Eqs. (35) or (36). The sheath thickness is $\eta_D = (125 c^3_3 \tilde{U}^{3/8})^{1/4}$ for the model of rigid spheres and $\eta_D = (9 \tilde{U}^{7/8})^{1/4}$ for Maxwell molecules. In dimensional variables, this solution reads

$$v_i = \left( \frac{9 e^2 J_i^4 \gamma^2}{4 e_0^2 m_i} \right)^{1/6}, \quad \Phi = \left( \frac{81 e^4 m_i J_i^2 \gamma^4}{16 c^4_3 e_0^4 k c_1^2} \right)^{1/6},$$

$$kT_i = \left( \frac{9 c^4_3 e^2 k m_i^2 c_1^2 J_i^2 \gamma^2}{32 e_0^4 k c_1^2} \right)^{1/3},$$

$$\phi = \left( \frac{81 e^4 m_i J_i^2 \gamma^2}{16 c^4_3 e_0^4 k c_1^2} \right)^{1/3}, \quad y_D = \left( \frac{c_6 k e_0 c_1^2 U^6}{e_0^4 m_i J_i^4} \right)^{1/10}.$$  \hspace{1cm} (39)

for the model of rigid spheres and
It is of interest that expressions for the ion velocity, the electric field, the potential, and the sheath thickness in this equation involve $\bar{\lambda}_i$ and $m_i$ only through combination $\bar{\lambda}_i/m_i$. A reason is that the ion mobility becomes independent of $T_i$ if written as $\mu_i = e\bar{\lambda}_i/m_i v_i$.

In Table I, orders of magnitude are shown of the ratio of the kinetic energy of directed motion and of thermal energy of the ions to the work of the sheath electric field. Also shown is order of magnitude of the ratio of the sheath thickness to the ion mean free path. The orders at $\bar{U} = O(1)$ are unity due to normalization; the orders at small and large $\bar{U}$ follow from the above asymptotic results.

One can see that $\bar{\lambda}_i \gg y_D$ in the case $\bar{U} \ll 1$ and $\bar{\lambda}_i \ll y_D$ in the case $\bar{U} \gg 1$, in accord to the above discussion. Orders of magnitude of the kinetic and thermal energies may be understood as follows. Ions enter the sheath with zero directed velocity and zero temperature in the framework of the considered model. Inside the sheath, the ions are accelerated by the sheath electric field and are heated due to conversion of kinetic energy of the directed motion into thermal energy in elastic collisions with the atoms. In the case of small $\bar{U}$, the kinetic energy equals the work of the electric field. The conversion of kinetic energy into thermal energy is poorly effective due to a small number of collisions in the sheath, and the ratio of the thermal energy to the kinetic energy (or to the work of the sheath electric field) is in this case of the order of $y_D/\bar{\lambda}_i \ll 1$. In the case of large $\bar{U}$, the conversion is effective and the kinetic and thermal energies are comparable. Both energies are in this case of the order of the work of the electric field over the ion mean free path and the ratio of these energies to the work of the electric field over the sheath is of the order of $\bar{\lambda}_i/y_D \ll 1$.

Another interpretation of the quantities shown in Table I may be given with a reference to the ion momentum equation, Eq. (2): $m_i v_i^2/eU$ characterizes the order of magnitude of the ratio of the ion inertia term to the electric field term, $kT_i/eU$ characterizes the order of magnitude of the ratio of the pressure gradient term to the electric field term, and $y_D/\bar{\lambda}_i$ characterizes the order of magnitude of the ratio of the collisional term to the ion inertia term. It follows that the ion pressure gradient term is minor at both small and large $\bar{U}$ while the ion inertia term is small at large $\bar{U}$, in accord to the above discussion.

The fact that the ion pressure gradient produces a negligible effect on the ion motion at both small and large $\bar{U}$ suggests that an (at least qualitatively) reasonable approximation may be obtained by dropping this term. In the framework of such an approximation, equations in Eq. (25) for the model of Maxwell molecules are decoupled from Eq. (26) and may be integrated analytically. One gets the parametric solution

$$V = E + e^{-E} - 1, \quad \Phi = -\frac{E^3}{3} + \frac{E^2}{2} - 1 + Ee^{-E} + e^{-E},$$

(42)
V. NUMERICAL SOLUTION

In this section, results of numerical solution of problem Eqs. (20)–(22) are given. Results of computations for the model of rigid spheres are shown in Fig. 1. The dashed lines represent asymptotic solutions described by Eqs. (29), (32), and (35). As it should have been expected, the agreement between the numerical and respective asymptotic results at small and large \( h \) is good.

In Fig. 2, results of computations for the model of Maxwell molecules are shown. The dashed lines represent the analytic solution described by Eqs. (42)–(44). The numerical and analytical results are close at all \( \eta \), which is an indication that the ion pressure gradient term is minor not only at small and large \( \eta \), but at finite \( \eta \) as well.

In Figs. 3 and 4, ratios of the kinetic energy of directed motion and of thermal energy of the ions at the cathode surface to the work performed by the sheath electric field are shown as functions of the normalized sheath voltage. (Index \( w \) means that respective quantity is evaluated at the cathode surface.) Dashed lines represent first-approximation asymptotic expressions which follow from Eqs. (29), (32), (35), and (36). One can see that the range of \( \tilde{U} \) in which a transition occurs from a collisionless to collision-dominated sheath is much wider for the model of Maxwell molecules than for the model of rigid spheres. The reason is clear from Table I: ratio \( \eta D / \lambda_i \) tends to zero at \( \tilde{U} \to 0 \) and to infinity at \( \tilde{U} \to \infty \).
much faster for the model of rigid spheres than for Maxwell molecules. One can see from Fig. 4 that \( kT_{\text{w}}/eU \) is rather small at all \( \tilde{U} \) for both models (does not exceed approximately 0.1), which conforms to the above conclusion on smallness of the pressure gradient term.

In Fig. 5, normalized values of the sheath thickness, of the electric field at the cathode surface, and of the ion energy flux to the cathode surface, \( q = 2.5\theta_{\text{w}} + 0.5V_{\text{w}}^2 \), are shown as functions of the normalized sheath voltage. One can see that the sheath thickness and the electric field at the surface for the two models are rather close, while the ion energy flux is essentially different except at low \( \tilde{U} \). This may be understood as follows. At low \( \tilde{U} \), functions \( \eta_{\text{D}}(\tilde{U}) \) and \( E_{\text{w}}(\tilde{U}) \) to a first approximation do not depend on the model of ion-atom interaction, which follows from the abovementioned fact that Eq. (28) is independent of the model of ion-atom interaction. Since \( M_{\text{ia}} \rightarrow \infty \) as \( \tilde{U} \rightarrow 0 \), function \( q(\tilde{U}) \) at small \( \tilde{U} \) to a first approximation equals \( V^2/2 \) and also does not depend on the model of ion-atom interaction. At large \( \tilde{U} \), functions \( \eta_{\text{D}}(\tilde{U}) \) and \( E_{\text{w}}(\tilde{U}) \) are related by formula \( E_{\text{w}} = (5/3)(\tilde{U}/\eta_{\text{D}}) \) for the model of rigid spheres and by \( E_{\text{w}} = (3/2)(\tilde{U}/\eta_{\text{D}}) \) for the model of Maxwell molecules. Since the numerical factors in these formulas are close, one should conclude that if functions \( \eta_{\text{D}}(\tilde{U}) \) for the two models are close, functions \( E_{\text{w}}(\tilde{U}) \) are close as well and vice versa. Ratios of functions \( \eta_{\text{D}}(\tilde{U}) \) and \( q(\tilde{U}) \) for the model of Maxwell molecules to the respective functions for the model of rigid spheres can be evaluated by means of Eqs. (33) and (34) and are equal, respectively,

\[
\frac{3^{5/3}U^{1/15}}{2^{7/5}5^{3/5}c_{\text{w}}^{2/5}} \approx 0.84\tilde{U}^{1/15}, \quad \frac{2^{22/4}U^{4/15}}{3^{13/2}2^{5/2}c_{\text{w}}^{7/5}(5c_{\text{w}}^{-2} + 2)} \approx 2.6\tilde{U}^{4/15}. \tag{45}
\]

For example, for \( \tilde{U} = 50 \) the respective values are 1.09 and 7.5, in accord to the tendency seen on the graph.

VI. CONCLUSIONS

If the voltage drop in a near-cathode plasma region exceeds substantially \( kT_{\text{w}}/e \), the electron density in the bulk of the space-charge sheath is much smaller than the ion density and the sheath may be to a first approximation considered as electron-free. In this work, a theory of such a sheath is developed on the basis of a fluid description of the ion motion for arbitrary relation between the sheath thickness and the mean free path for ion-atom collisions and a variable ion temperature.

What gives the fluid description of the ion motion its usefulness is the fact that results of a fluid model give usually a smooth transition from a free-fall model to a collision-dominated model. Therefore, special attention has been paid to investigation of limit cases of low \( \tilde{U} \), when the model gives the well-known Child–Langmuir solution describing a free-fall sheath formed by ions entering the sheath with zero velocity, and of high \( \tilde{U} \), when the model results in a solution describing a collision-dominated mobility-controlled sheath.

Heating of ions in the sheath is at low \( \tilde{U} \) negligible due to smallness of number of collisions. At high \( \tilde{U} \), the ion thermal energy is of the order of the work of the electric field over the ion mean free path and is therefore much smaller than the work of the electric field over the sheath. As a consequence, the pressure gradient term of the ion momentum equation is asymptotically small as compared to the electric field term at low and high \( \tilde{U} \). This feature of the model conforms to the fact that the ions constitute a monoenergetic beam in a collisionless electron-free sheath and the ion momentum equation in such a sheath must assume the form of the equation of motion of an ion. On the other hand, the above feature conforms to the fact that the ion motion in a collision-dominated electron-free sheath is mobility-controlled (i.e., the diffusion velocity of the ions is much smaller than the drift velocity; e.g., Refs. 7, 20–22).

Numerical calculations show that the ion pressure gradient term does not affect appreciably the solution at all \( \tilde{U} \), including those of the order unity. The range of values of \( \tilde{U} \) in which the sheath is neither collisionless nor collision-dominated is much larger for the model of Maxwell molecules than for the model of rigid spheres.

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