THEORY OF ION KNUDSEN NEAR-ELECTRODE LAYER

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Abstract

A problem of the evaluation of ion distribution function in the continuum weakly ionized plasma at the distances of the order of the mean free path from the electrode surface is considered with regard to the influence of applied electric field. For the collision term being taken in the form of BGK-model, the kinetic equation is solved in the limiting cases of the weak and strong fields. On the ground of that solution the continuum boundary conditions at the absorbing and emitting electrodes are founded. It is shown that results which were obtained within the frames of the BGK-model in a case of emitting electrode retain their validity within the frames of more strict approach being founded on the full Boltzmann equation.

A problem of the evaluation of a charged particles distribution in the near-electrode regions of a weakly ionized plasma at the distances of the order of a mean free path from the electrode surface (in the Knudsen layer) is of great interest in connection with the deduction of the continuum boundary conditions for a particle concentration at the electrode surface.

In case the electric field influence in the Knudsen layer being negligible, the problem is elaborated elsewhere . In the near-electrode regions of the MHD-channel plasma this approach is invalid: the electric field influence is considerable. Present paper is devoted to the calculation of the ion distribution function in the Knudsen near-elecrode layer in a weakly ionized plasma taking into account the electric field influence.

1. Let us consider the problem of the evaluation of an ion distribution function in the Knudsen near-electrode layer in the weakly ionized high pressure plasma. It is supposed that the electrode is plane, convective motion is absent, the temperature of neutrals is constant and equals to the electrode temperature. The collisions between ions and electrons or other ions are neglected. It is supposed also that the thickness of the near-electrode space charge layer is much greater than the ion mean free path, so the electric field in the region considered can be taken to be constant. Following [1-4], we use the kinetic equation with collision integral in the form of BGK-model [5]

$$c\frac{\partial f}{\partial \eta} \pm \frac{1}{2\alpha}\frac{\partial f}{\partial c} + f = \frac{n}{u\sqrt{\pi}}\exp(-c^2), \qquad (1)$$

$$n = u\int_{-\infty}^{+\infty} f\,dc;$$

$$\eta = \frac{y}{\lambda}, \quad c = \frac{v_y}{u}, \quad \lambda = u\tau, \quad u = \sqrt{\frac{2kT}{m}},$$

$$\alpha = \frac{kT}{e|E|\lambda}$$

where $f = f(\eta, c)$, $n = n(\eta)$ are the ion distribution function and concentration; y is the distance from the electrode surface; v_y is the y-component of the particle velocity vector; the electric field intensity E is normal to the electrode surface, upper sign corresponds to its outward direction ($E > 0$), lower sign – to the inward direction ($E < 0$); τ is the BGK-model parameter; T is the neutrals temperature; m, e are the ion mass and its charge; k is the Boltzmann constant.

The above introduceo parameters λ, u are the characteristic mean free path and the characteristic thermal velocity evaluated with the ion temperature taken to be equal to the neutral temperature. α characterises the power of the applied electric field. The case $\alpha \gg 1$ corresponds to the weak field limit, $\alpha \ll 1$ – to the strong field limit [6].

The boundary conditions for equation (1) are as follows. In the case $E > 0$ (ions move from the electrode) the electrode surface emit ions with the Maxwellian distribution at the temperature T, the emission flux is j_{em}. In the case $E < 0$ (ions move to the electrode the electrode surface is fully absorbing and nonemitting. Thus we have

$$E > 0: \ f(0, c > 0) = 2 j_{em} \exp(-c^2)/u^2 \qquad (2)$$

$$E < 0: \ f(0, c > 0) = 0. \qquad (3)$$

Far away from the electrode the ion distribution function tends to the spatialtially isotropic solution of (1)

$$f_{\infty}(c) = n_{\infty}\alpha\exp(\alpha^2 \mp 2\alpha c)\operatorname{erfc}(\alpha \mp c)/u, \quad (4)$$

where n_{∞} – the ion concentration in the undisturbed region.

Thus the formulation of the problem is completed. If α being arbitrary, to obtain the analitic solution of the problem is difficult. The presented work purpose is to analyze solution in the weak and strong field limits by matched asymptotic expansions technique[7].

The total ion flux is introduced as follows

$$j = u^2\int_{-\infty}^{+\infty} c f\, dc$$

Integrating (4) gives the expression for the total flux in terms of n_{∞}, and also the expression for the mobility of ions

$$j = \pm\, u n_{\infty}/(2\alpha), \quad (5)$$

$$\mu = e\tau/m.$$

2. Let us consider the case of ions moving from the emitting surface ($E > 0$).

In the weak field limit the solution of the problem is obvious: the distribution function throughout the region under consideration is Maxwellian

$$f = 2j_{em}\exp(-c^2)/u^2.$$

The relationship between the concentration in the undisturbed region and the emission flux is

$$n = 2\sqrt{\pi}\, j_{em}/u\ . \quad (6)$$

In the strong field limit it is necessary to consider two asymptotic expansions of the concentration, which are valid at the distances $\eta = O(\alpha)$ and $\eta = O(\alpha^{-1})$ from the electrode surface. These expansions have the form

$$n(\eta;\alpha) = n_1(\eta_1) + \cdots, \quad \eta_1 = \eta/\alpha \geqslant 0; \quad (7)$$

$$n(\eta;\alpha) = \alpha n_2(\eta_2) + \cdots, \quad \eta_2 = \eta\alpha > 0; \quad (8)$$

To describe the distribution function it is necessary to consider five asymptotic expansions, their regions of validity are shown schematically on fig. 1.

The expansion being valid in the region $\eta = O(\alpha)$, $c = O(1)$, $c > \sqrt{\eta_1}$ (region I on fig. 1) is

$$f(\eta, c;\alpha) = f_1(\eta_1, c) + \cdots,$$

$$\eta_1 \geqslant 0,\ c > \sqrt{\eta_1}\ . \quad (9)$$

For f_1 we get the problem

$$c\frac{\partial f_1}{\partial \eta_1} + \frac{1}{2}\frac{\partial f_1}{\partial c} = 0\ ;\ f_1(0,c) = 2j_{em}e^{-c^2}/u^2.$$

The solution is

$$f_1 = 2j_{em}\exp(\eta_1 - c^2)/u^2.$$

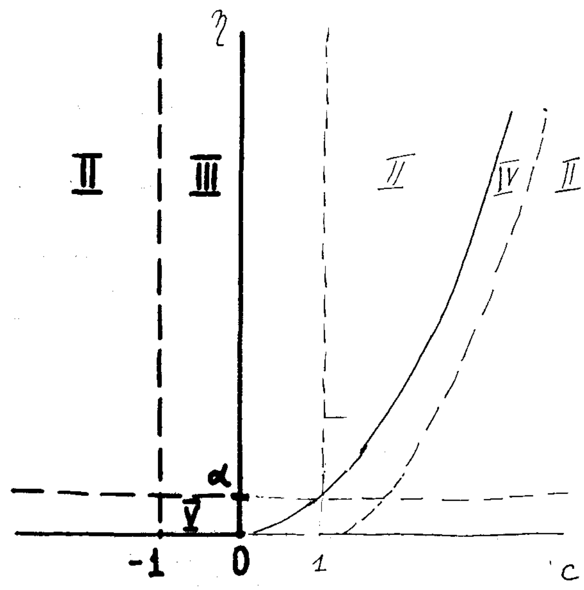

Fig. 1.

The expansion being valid in the region $\eta = O(\alpha^{-1})$, $|c| = O(\alpha^{-1})$ (region II on fig. 1) is

$$f(\eta, c;\alpha) = \alpha^2 + f_2(\eta_2, c_2) +$$

$$\eta_2 > 0,\ c_2 = \alpha c,\ c_2 \neq 0,\ c_2 \neq \sqrt{\eta_2}\ . \quad (10)$$

For f_2 we get the problem

$$c_2\frac{\partial f_2}{\partial \eta_2} + \frac{1}{2}\frac{\partial f_2}{\partial c_2} + f_2 = 0;\ f_2(0, c_2 > 0) = 0,$$

$$f_2(\infty, c_2) = \frac{2n_2}{u}\exp(-2c_2)\theta(c_2),$$

The solution is

$$f_2 = \begin{cases} 0, & c_2 < 0 \\ F_1(\eta_2 - c_2^2)\exp(-2c_2), & 0 < c_2 < \sqrt{\eta_2} \\ 0, & c_2 > \sqrt{\eta_2} \end{cases}$$

where F_1, F_2 – unknown functions, $F_1(\infty) = 2n_{2\infty}/u$.

The expansion being valid in the region $\eta = O(\alpha^{-1})$, $|c| = O(1)$ (region III on fig. 1)

$$f(\eta, c;\alpha) = \alpha^2 f_3(\eta_2, c) + \cdots$$

$$\eta_2 > 0,\ |c| \geqslant 0 \quad (11)$$

For f_3 we get the problem

$$\frac{1}{2}\frac{\partial f_3}{\partial c} = \frac{n_2}{u\sqrt{\pi}}\exp(-c^2),\ f_3(\infty, c) = \frac{n_{2\infty}}{u}\operatorname{erfc}(-c),$$

$$f_3(\eta_2, \infty) = F_1(\eta_2),\ f_3(\eta_2, -\infty) = 0,$$

For this problem can be solved F_1 is to be equal to

$$F_1 = 2n_2/u.$$

The solution is

$$f_3 = n_2 \,\mathrm{erfc}(-c)/u.$$

The expansion being valid in the vicinity of the line $c = \sqrt{\eta/\alpha}$, $\eta = O(\alpha^{-1})$ (region **IV** on fig. 1) is

$$f(\eta, c; \alpha) = f_4(\eta_2, c_4) + \cdots,$$

$$\eta_2 > 0, \quad c_4 = (c - \sqrt{\eta/\alpha})/\alpha > 0. \qquad (12)$$

For f_4 we get the equation

$$\sqrt{\eta_2}\,\frac{\partial f_4}{\partial \eta_2} - \frac{c_4}{2\sqrt{\eta_2}}\,\frac{\partial f_4}{\partial c_4} + f_4 = 0.$$

The solution satisfiing the condition of matching with the solution in region is

$$f_4 = 2j_{em} \exp(-2c_4\sqrt{\eta_2}) \exp(-2\sqrt{\eta_2})/u^2.$$

The expansion being valid in the region $\eta = O(\alpha)$, $|c| = O(1)$, $c < \sqrt{\eta_1}$ (region **I** on fig. 1) is

$$f(\eta, c; \alpha) = \alpha f_5(\eta_1, c) + \cdots, \quad \eta_1 > 0, \; c < \sqrt{\eta_1}. \qquad (13)$$

The substitution of expansions (7), (9) and (13) into the second equation (1) yields an explicit expression for n_1

$$n_1 = \sqrt{\pi}\, j_{em} \exp(\eta_1)\, \mathrm{erfc}\sqrt{\eta_1}/u;$$

$$\eta_1 \to \infty: \; n_1 = j_{em}/(u\sqrt{\eta_1}) + \cdots.$$

For f_5 we get the problem

$$c\,\frac{\partial f_5}{\partial \eta_1} + \frac{1}{2}\frac{\partial f_5}{\partial c} = \frac{n_1}{u\sqrt{\pi}}\exp(-c^2); \quad f_5(\eta_1, -\infty) = 0,$$

$$\eta_1 \to \infty: \; f_5 = j_{em}\,\mathrm{erfc}(-c)/(u^2\sqrt{\eta_1}) + \cdots.$$

The solution is

$$f_5 = (2j_{em}/u^2)\exp(\eta_1 - c^2)\int_{-\infty}^{c} \mathrm{erfc}\sqrt{\eta_1 - c^2 + x^2}\, dx.$$

The substitution of expansions (8), (10), (11), (12) into the second equation (1) yields an integral equation for n , which is solved by use of the Laplace transform. From that solution we derive the relationship between the concentration in the undisturbed region and the emission flux

$$n_\infty = j_{em}/v_d. \qquad (14)$$

The physics of the solutions obtained is clear. In the weak field limit the distribution function in the first order approximation throughout the halfspace considered is Maxwellian. The relationship (6) coincides with the results [8,9] (with neglect of low order terms in the references mentioned).

In the strong field limit the solution obtained is abruptly anysotropic and is characterized by two length scales. On the scale of diffuse length $l_d = kT/(eE)$ the velocities, which ions gained in the field, are of the order of thermal velocitv u. The second length scale $\lambda_\lambda = \lambda/(2\alpha) = V_d t$ is the ion mean free path evaluated with the drift velocity. The ions in the region IV did not collide yet after emission from the surface; it is naturally to refer to this ions as the ion beam. If increasing the distance from the electrode surface velocity dispersion is decreased as $y^{-1/2}$ (it is naturally to refer to this phenomenon as selffocusing). Ions leaving the beam due to the scattering transit to the thermal velocity region III, and then from this region - to the region $0 < c < \sqrt{\eta/\alpha}$.

The physics of the relationship (14) is clear: all emitted ions leave the near-electrode region under the action of the applied field. The relationships (6), (14) are the continuum boundary conditions on the emitting surface. From these relationships a simple interpolation may be deduced for this boundary condition at arbitrary values of E

$$n_\infty = j_{em}/(v_d + \sqrt{kT/(2\pi m)}) \qquad (15)$$

The graph of the quantity $n_{\lambda t} V_d / j_{em}$ being defined from (15) for ions K^+ in nitrogen at T = 700 K is shown on fig. 2 by solid line. The weak field limit corresponds to the electric field intensity $E < 10^6$ V/sm, the strong field limit - $E > 10^6$ V/sm.

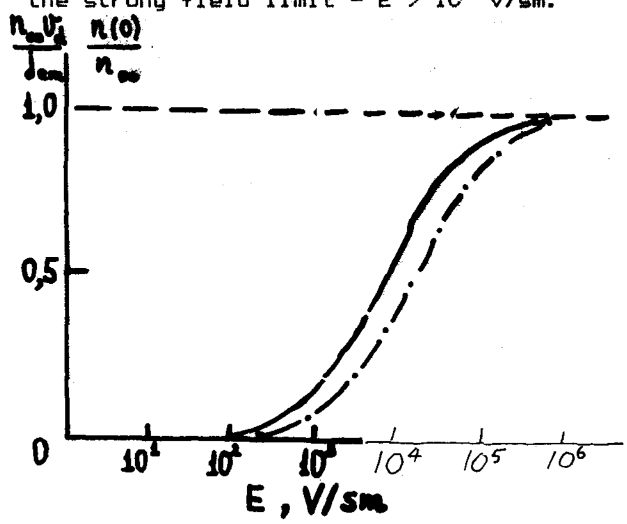

Fig. 2.

3. In the case of the ions moving towards we take the total ion flux as a given quantity. At first let us consider the weak field limit. The asymptotic expansion being valid in the Knudsen near-electrode layer is

$$f(\eta, c; \alpha) = f_6(\eta, c) + \cdots, \quad \eta > 0 \qquad (16)$$

Substituting this expansion into (1) and taking into account (3) we obtain the problem for

$$c\,\frac{\partial f_6}{\partial \eta} + f_6 = \frac{n_6}{u\sqrt{\pi}}\exp(-c^2), \quad n_6 = u\int_{-\infty}^{+\infty} f_6\, dc;$$

$$f_6(0, c > 0) = 0$$

This problem is wellknown as Milne problem. The solution of the problem was obtained, for example, in [1]. In particular the function $n_6(\eta)$ is given by the formula [1]

$$n_6 = C_1(\eta + l_0 - n_2(\eta)),$$

where $l_0 = 1{,}016$, expression for the function $n_2(\eta)$ was obtained in [1] (note, that $n_2(0) = l_0 - 1/\sqrt{2}$, $n_2(\infty) = 0$), C_1, C_2 are the integration constants.

If $\eta \to \infty$ the asymptotic of the function f_6 is [2]

$$f_6 = \frac{C_1 \eta}{u\sqrt{\pi}} \exp(-c^2) + \cdots . \qquad (17)$$

This asymptotic does not obey the condition (4). Therefore it is nesessary to consider intermediate asymptotic expansion, which is valid between the Knudsen near-electrode layer and the undisturbed region. This expansion is

$$f(\eta, c; \alpha) = \alpha f_7(\eta_7, c) + \cdots, \eta_7 = \eta/\alpha > 0. \quad (18)$$

The substitution of this expansion into (1) yields

$$f_7 = \frac{n_7}{u\sqrt{\pi}} \exp(-c^2), \; n_7 = u\int_{-\infty}^{+\infty} f_7 \, dc. \qquad (19)$$

Multipliing (1) by c and then integrating we obtain the equation for the total ion flux. Using the expression for f_7 we get

$$j = -\frac{u}{2}\left[\frac{dn_7}{d\eta_7} + n_7\right]. \qquad (20)$$

Obviously this relationship is the usual transport equation.

The solution of (20) in account of (5) is given by the formula

$$n_7(\eta_7) = n_{7\infty}(1 + C_2 \exp(-\eta_7)), \qquad (21)$$
$$n_{7\infty} = -\frac{2j}{u} .$$

The condition of matching of expansions (16) and (18) gives the boundary condition for f_7 at $\eta_7 = 0$. Taking into account relations (17), (19), (21) we obtain

$$C_1 = n_{7\infty}, \; C_2 = -1.$$

Obviously the function f_7 satisfies the condition (4).

Thus the solution of the problem in the weak field limit is obtained. The relationship between the ion concentration at the electrode surface and the total ion flux is

$$n(0) = -\sqrt{2}\, j/u . \qquad (22)$$

In the strong field limit the distribution function throughout the halfspace considered except for the region $\eta = O(\alpha)$, $|c| = O(1)$ is spatially uniform, the velocity dependence is described by the formula (4). In the exepted region the asymptotic expansion of the distribution function is valid

$$f(\eta, c; \alpha) = \alpha^2 f_8(\eta_8, c) + \cdots, \eta_8 = \eta/\alpha > 0. \quad (23)$$

For f_8 we get the problem

$$c\frac{\partial f_2}{\partial \eta_8} - \frac{1}{2}\frac{\partial f_8}{\partial c} = \frac{-2j}{u^2\sqrt{2\pi}} \exp(-c^2); \; f_8(0, c>0) = 0$$

$$f_8(\infty, c) = \frac{2j}{u^2}\,\mathrm{erfc}(c), \; f_8(\eta_8, \infty) = 0, \; f_8(\eta_8 \to \infty) = \frac{-4j}{u^2}$$

The solution is

$$f_8 = -\frac{2j}{u^2}\left(\mathrm{erfc}(c) - \mathrm{erfc}\sqrt{\eta_8 + c^2}\right).$$

The relationship between the ion concentration at the electrode surface and the total ion flux is

$$n(0) = j/v_d . \qquad (24)$$

Let us consider the physics of the solutions obtained. In the weak field limit the scale of disturbed region is determined by the diffuse length l_d. This length is considerably more than the mean free path. The distribution function alternates between the isotropic in the undisturbed region and abruptly anisotropic at the electrode surface. The ion concentration at the surface is considerably less than the concentration in the undisturbed region.

In the strong field limit the distribution function is characterized by two velocity scales: u is the thermal velocity and v_d is the drift velocity. On the scale v_d the distribution function is abruptly anisotropic: the number of ions moving in the direction of the field is considerably more then the number of ions moving in the opposite direction. Therefore on the scale v_d the distribution function is not disturbed by the electrode. On the scale u the number of ions moving in the direction of the field and the number of ions moving in the opposite direction are comparable. Therefore on this scale the distribution function is disturbed by the electrode, the length scale of disturbed region is determined again by the diffuse length l_d. Since the contribution to the concentration from the velocity region v_d is greater than the contribution from the region u, the ion concentration throughout the near-electrode region in the first approximation is equal to the ion concentration in the undisturbed region.

As in the case E > 0 from the (22),(24) it is possible to deduce a simple interpolation

$$n(0) = -j/\left(|v_d| + \sqrt{kT/m}\right) \qquad (25)$$

This relation may be used as the boundary condition at the electrode surface y = 0 for the continuum transport equation. Note that within the frames of commonly used approach (with zero ion concentration at the electrode surface [10,11]) the continuum solution is physicaly invalid in the Knudsen layer [11]. In contrast to such an approach the use of the relationship (18) as the continuum boundary condition enables to obtain qualitively true (coinsiding with results of the kinetic consideration) concentration distribution in the both weak and strong field limits. The graph of the ratio of the concentration at the electrode surface to the concentration in the undisturbed region defined by the formula (18) is shown on fig. 2 by the dot-and-dashed line. The both weak and strong field limits correspond to a little more values of the electric field intensities in comparison with the case $E > 0$.

4. It should be noted that the validity of the above consideration is restricted to the BGK-model for collision integral in the Boltzmann equation. It can be shown, however, that some results obtained are still valid within the frames of the more strict analysis, being founded on the full Boltzmann equation. Let us consider, for example, the ion emission problem in the strong field limit. We take the dependence of the elastic scattering cross-section of ions on neutrals upon their relative velocity in the commonly used form $\sigma \sim v^{-\varkappa}$, $0 \leq \varkappa \leq 1$ [6]. As above, the strong field limit correspond to the small values of $\alpha = kT/(eE\lambda)$, where λ is characteristic ion mean free path evaluated with ion velocity taken to be equal to the thermal velocity u. As previously disturbed region may be characterized by two length scales. On the scale of l_α the ion velocities are of the order of the thermal velocity u, two asymptotic expansions are valid for distribution function. The first expansion describes the distribution function in the region $v_y > \sqrt{2eEy/m}$ (region I on fig. 1). In that region the value of collision integral in the Boltzmann equation is negligible in comparison with the transport terms, the solution (9) is valid, the dependence upon v_x and v_z is given by the Maxvellian factor $(\pi u^2)^{-1} \exp(-(v_x^2 + v_z^2)/u^2)$. The second expansion is valid in the region $v_y < \sqrt{2eEy/m}$ (region II on fig. 1). Considerable contribution is given by the ions scattered from the region I, the distribution function is of the order of $O(\alpha\, j_{em}/u^2)$.

On the scale $\lambda_\alpha = \lambda(2\alpha)^{\varkappa/(\varkappa-2)}$ the ion velocities are of the order of the drift velocity $v_\alpha = u(2\alpha)^{1/(\varkappa-2)}$ and considerably more than the velocities of neutrals. Hence the neutrals may be considered as stationary. The distribution function is described by two asymptotic expansions. The first describes the beam and in the first approximation has the appearance (dependence upon v_x, v_z is given by the Maxvellian factor)

$$f = 2\frac{j_{em}}{u^2} \exp\left[-\frac{m}{kT}\sqrt{\frac{2eEy}{m}}\left(v_y - \sqrt{\frac{2eEy}{m}}\right) - \frac{2}{2-\varkappa}\left(\frac{kT}{eE\lambda}\right)^{\varkappa/2}\left(\frac{y}{\lambda}\right)^{(2-\varkappa)/2}\right].$$

The second expansion is valid in the region $|\vec{v}| < \sqrt{2eEy/m}$ and describes ions, which leave the beam due to the scattering. The order of distribution function is $O(\alpha^{2/(2-\varkappa)} j_{em}/u^2)$. If collided with stationary neutral, the ion cannot gain energy. Therefore the value of the distribution function in the region $|\vec{v}| > \sqrt{2eEy/m}$ in the first approximation is equal to zero.

Obviously the formula (14) in the first approximation still holds due to the simple physical reason: emitted ion, losing the energy in collisions with neutrals, cannot return to the electrode. The asymptotic formula for the weak field limit (6) was obtained without the use of BGK-model. Thus the validity of the relationship (15) does not connect with the use of BGK-model.

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