Investigation of Stability of Current Transfer to Thermionic Cathodes

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Abstract: Spectra of perturbations of 3D steady-state current transfer to thermionic cathodes of a high-pressure argon arc have been computed in the framework of the model of nonlinear surface heating, but were uncompleted. Spectra of perturbations with a different symmetry than that of the 3D steady-state mode itself were lacking. Presently, the difficulty in calculation of these perturbations by means of COMSOL Multiphysics 3.4 has been overcome and a modeling of a complete spectrum is possible. Results obtained conform to the analytical theory.

Keywords: eigenvalue problem, stability, 3D steady-state modes, perturbation modes.

1. Introduction

Interaction of high-pressure arc plasmas with thermionic cathodes is a challenging issue of high scientific interest and technological importance. Current transfer from high-pressure arc plasmas to thermionic cathodes may occur in a diffuse mode, when the current is distributed over the front surface of the cathode in a more or less uniform way, or in a spot mode, when most of the current is localized in one or more small areas (cathode spots). The diffuse mode is favorable for operation of cathodes of high-pressure devices, however it is difficult to be realized, because under many conditions this mode is unstable and one of the spots modes appears. Recently a self-consistent theory and modeling methods have started to emerge. Still, some important questions are far from being answered. Stability of different modes of steady-state current transfer from high-pressure arc plasmas to thermionic cathodes is one of such questions. This information is important for engineering practice, since one needs to know which of the modes are stable in conditions of interest.

The present work represents a continuation of a numerical investigation of stability started in [1]. In [1], only the part of the spectrum that is associated with even eigenfunctions was calculated. The part of the spectrum that is associated with odd eigenfunctions was analyzed on the basis of theoretical results of [2]. Presently, the difficulty in calculation of odd perturbations by means of COMSOL Multiphysics 3.4 has been overcome and a modeling of a complete spectrum became possible. Such modeling is reported in this paper.

2. Model and numerics

2.1 Equations and boundary conditions

Modeling of the present work is based on the model of nonlinear surface heating, which has been by now widely recognized as an adequate tool of simulation of interaction of high-pressure arc plasmas with thermionic cathodes; see review [3] and references therein. The temperature distribution inside the cathode is governed by the non-stationary equation of heat conduction:

$$\rho(T)\, c_p(T)\, \frac{\partial T}{\partial t} = \nabla \cdot \left(\kappa(T)\, \nabla T\right), \qquad (1)$$

where κ, ρ, and c_p are thermal conductivity, density, and specific heat of the cathode material (known functions of the temperature T) and t is time. Joule heat production inside the cathode is neglected.

The base Γ_c of the cathode is maintained at a fixed temperature T_c by external cooling. The rest of the cathode surface, Γ_h, is in contact with the plasma or the cold gas and exchanges energy with it. The boundary conditions read

$$\Gamma_c : T = T_c; \qquad \Gamma_h : \kappa(T)\, \frac{\partial T}{\partial n} = q(T,U). \quad 2)$$

Here n is the direction locally orthogonal to the cathode surface and oriented outward and q is the density of the energy flux to the cathode surface from the arc plasma or the cold gas (a known function of the local cathode temperature T and the voltage drop U across the near-cathode layer.)

The total electric current I to the cathode surface (the arc current) may be evaluated in terms of a distribution of the cathode surface temperature and of the value of U by means of the formula

$$I = \int_{\Gamma_h} j(T,U)\,dS, \tag{3}$$

where $j = j(T,U)$ is the density of electric current to the current-collecting part of the cathode surface. A relation between U and I is given by the equation of external circuit

$$U + I\Omega = \varepsilon, \tag{4}$$

where Ω is the external resistance (ballast) and ε the electromotive force.

Densities of the energy flux and of the electric current to the cathode surface, q and j, are governed by equations describing the near-cathode layer in a high-pressure arc plasma and in the present analysis are treated as known functions of the local surface temperature and of the voltage drop across the near-cathode layer: $q = q(T,U)$ and $j = j(T,U)$.

2.2 Numerical solution

A method of investigation of stability of steady states employed in [1] was as follows. Equation (1) has been solved numerically by means of COMSOL Multiphysics, version 3.4, through a heat transfer application mode, using the stationary and eigenvalue solvers. A weak point of this approach is that it does not allow different boundary conditions for a steady-state solution and perturbations. This can be explained as follows.

Modeling results presented in this study refer to cathodes in the form of a right circular cylinder. Steady-state temperature distributions on axially symmetric cathodes may be axially symmetric or 3D. Axially symmetric distributions correspond to the diffuse mode and to modes with axially symmetric spot systems, 3D distributions correspond to modes with 3D spot systems. Let us restrict the consideration to steady-state temperature distributions that are either axially symmetric or 3D with planar symmetry. Let us introduce Cartesian coordinates (x,y,z) with the origin at the center of the front surface of the cathode and the z-axis directed along the axis inside the cathode, in such a way that the steady-state temperature distribution be even with respect to y.

While numerically calculating a steady-state 3D distribution on an axially symmetric cathode, one must impose an additional condition in order to specify azimuthal position of the 3D spot system, and thus to ensure convergence of iterations. This can be achieved by restricting the calculation domain to half of the cathode, $y \geq 0$, and imposing the symmetry condition at the plane $y = 0$: $\partial T_0/\partial y = 0$, where the index 0 is attributed to the steady state. Of course, this approach allows one not only to fix the azimuthal position of a 3D spot system but also to save RAM and CPU time.

There is no problem in restricting the calculation domain to half of the cathode also while stability of steady-state solutions is investigated. However, the use of COMSOL Multiphysics implies that the symmetry boundary condition is imposed also on perturbations: $\partial T_1/\partial y = 0$ at $y = 0$, where index 1 is attributed to the amplitude (time-independent factor) of perturbations. In other words, COMSOL Multiphysics allows one to study stability against axially symmetric perturbations and 3D perturbations that are even with respect to y, but not against 3D perturbations that are odd with respect to y. This does not pose a problem while stability of axially symmetric steady states is investigated: 3D even and odd perturbations of an axially symmetric steady state are identical to the accuracy of a rotation and are therefore associated with the same eigenvalue (which is, consequently, doubly degenerated), hence an account of 3D odd perturbations will not change conclusions on stability. The situation is different as far as stability of even 3D steady states is concerned: odd and even perturbations of even 3D steady states are essentially different and are therefore associated with different eigenvalues, and eigenvalues associated with odd perturbations cannot be computed by means of COMSOL Multiphysics using only a heat transfer application mode.

To overcome this difficulty, the following procedure was applied: a solution to the problem (1)-(4) is sought as sum of a steady-state solution and a small perturbation with the exponential time dependence

$$T(\vec{r},t) = T_0(\vec{r}) + e^{\lambda t} T_1(\vec{r}) + ...,$$
$$U(t) = U_0 + e^{\lambda t} U_1 + ..., \qquad (5)$$
$$I(t) = I_0 + e^{\lambda t} I_1 + ...$$

Here $\vec{r}$ is the space vector and λ is the growth increment of the perturbations. Substituting these expansions into Eqs. (1)-(4), linearizing and equating linear terms, one obtains

$$\rho(T_0) c_p(T_0) \lambda T_1 = \nabla \cdot \left(\frac{d\kappa}{dT}(T_0) T_1 \nabla T_0 + \kappa(T_0) \nabla T_1 \right), \qquad (6)$$

$$\Gamma_c : T_1 = 0, \qquad (7)$$

$$\Gamma_h : \frac{d\kappa}{dT}(T_0) T_1 \frac{\partial T_0}{\partial n} + \kappa(T_0) \frac{\partial T_1}{\partial n} = = \frac{\partial q}{\partial T}(T_0, U_0) T_1 + \frac{\partial q}{\partial U}(T_0, U_0) U_1, \qquad (8)$$

$$I_1 = \int_{\Gamma_h} \left(\frac{\partial j}{\partial T}(T_0, U_0) T_1 + \frac{\partial j}{\partial U}(T_0, U_0) U_1 \right) dS, \qquad (9)$$

$$U_1 + I_1 \Omega = 0. \qquad (10)$$

Here $\rho(T_0)$, $((\partial q)/(\partial T))(T_0, U_0)$ etc are evaluated in terms of the temperature distribution of the steady state, T_0, and of the near-cathode voltage drop U_0 corresponding to the steady state.

Eq. (6) with the boundary conditions (7)-(10) represents a closed linear eigenvalue problem for the function T_1 and the eigenvalue λ. By means of solving this problem, one will determine a set of eigenvalues λ (spectrum) for every stationary state of interest.

To implement this approach in COMSOL Multiphysics, a PDE mode that describes the problem (6)-(10) must be added to the heat transfer application mode [or to the PDE mode that describes the problem (1)-(4)]. The even perturbations spectrum can be calculated through the system of equations that computes the steady-state modes or through the added PDE mode with $\partial T_1/\partial y = 0$ at the plane $y = 0$. The odd perturbations spectrum is calculated through the added PDE mode with $T_1 = 0$ at the plane $y = 0$.

3. Numerical results

Numerical calculations reported in this work have been performed for a tungsten cathode of radius $R = 2$ mm and height $h = 10$ mm, and an argon plasma at the pressure of 1 bar. Data on thermal conductivity and heat capacity of tungsten have been taken from [4] and [5], respectively. The density of tungsten equals 19,250 kgm^{-3} and the value of 4.55 eV was assumed for the work function of tungsten. The cooling temperature T_c was set equal to 293 K. Functions $q = q(T,U)$ and $j = j(T,U)$ have been calculated by solving equations describing the near-cathode layer in a high-pressure plasma which are summarized in [6], see also the Internet site [7].

Table 1: Period T and signal of the increments of the six perturbations with biggest values of the increment for steady-state modes with one, two, three and four spots at the edge of the frontal surface of the cathode, at the bifurcation point (b.p.) and beyond it. υ: number of spots at the edge of the frontal surface of the cathode.

υ	T	Signal of increment (at the b.p.)	Signal of increment (beyond b.p.)
1	2π	0, -, -, -	0, -, -, -, -, -
	π	-	
	2π /3	-	
2	2π	+,-, -, -	+, -, -, -, -
	π	0	0
	2π /3	-	
3	2π	+, -, -	+, +, -, -, -
	π	+	
	2π /3	0	0
	π/4	-	
4	2π	+, -, -	+, +, -, -
	π	+	+
	2π /3	+	
	π/4	0	0

Stability of 3D steady-state modes against odd perturbations were performed for modes with $\upsilon = 1, 2, 3, 4$ spots at the edge of the frontal surface of the cathode. Odd perturbations of these steady-states are periodic with respect to the azimuthal angle with period T. Table 1 shows the period T and the signal of the increment of the six odd perturbations with biggest values of the increment at the 'initial' state and at all the states beyond the bifurcation point for the above steady-state modes. (It should be emphasized that 3D spot modes at the 'initial state' are axially symmetric, and in the following, conclusions refer to when the 3D spot modes are effectively 3D.) If the increment of a perturbation is null, the state is neutrally stable against this perturbation; if the signal of the increment is positive, the state is unstable against this perturbation; and if the signal of the increment is negative, the state is stable against this perturbation.

One can see from table 1 that the 3D steady-state mode with one spot at the edge of the frontal surface of the cathode possesses an odd perturbation with period 2π that always has a null increment. All the other odd perturbations always have negative increments. However, their periods are not always the same: at the bifurcation point ('initial state'), all periods are permitted, but beyond the bifurcation point only the period 2π is permitted. Therefore, the 3D steady-state mode with one spot at the edge of the frontal surface of the cathode is neutrally stable against one odd perturbation with period 2π and stable against all the other odd perturbations.

The 3D steady-state mode with two spots at the edge of the frontal surface of the cathode possesses: an odd perturbation with period 2π that always has positive increments; and an odd perturbation with period π that always has a null increment. All the other odd perturbations always have negative increments. However, their periods are not always the same: at the bifurcation point all periods are permitted, but beyond it only periods 2π and π are permitted. Therefore, the 3D steady-state mode with two spots at the edge of the frontal surface of the cathode is neutrally stable against one odd perturbation with period π, unstable against an odd perturbation with period 2π and stable against all the other odd perturbations.

The 3D steady-state mode with three spots at the edge of the frontal surface of the cathode possesses: an odd perturbation with period 2π that always has positive increments; an odd perturbation with period π at the bifurcation point and period 2π beyond it that always has positive increments; and an odd perturbation with period $2\pi/3$ that always has a null increment. All the other odd perturbations always have negative increments. However, their periods are not always the same: at the bifurcation point all periods are permitted, but beyond it only periods 2π and $2\pi/3$ are permitted. Therefore, the 3D steady-state mode with three spots at the edge of the frontal surface of the cathode is neutrally stable against one odd perturbation with period $2\pi/3$, unstable against two odd perturbations with period 2π and stable against all the other odd perturbations.

The steady-state mode with four spots at the edge of the frontal surface of the cathode possesses: two odd perturbations, one with period 2π and other with period π, that always have positive increments; an odd perturbation with period $2\pi/3$ at the bifurcation point and period 2π beyond it that always has positive increments; and an odd perturbation with period $\pi/4$ that always has a null increment. All the other odd perturbations always have negative increments. However, their periods are not always the same: at the bifurcation point all periods are permitted, but beyond it only periods 2π, π and $\pi/4$ are permitted. Therefore, the 3D steady-state mode with four spots at the edge of the frontal surface of the cathode is neutrally stable against one odd perturbation with period $\pi/4$, unstable against two odd perturbations with periods 2π and one odd perturbation with period π, and stable against all the other odd perturbations.

The above results may be summarized as follows. Odd perturbations do not change sign of their increment along 3D steady-state spot modes. Odd perturbations of a steady-state mode with υ spots at the edge of the front surface of the cathode are periodic with respect to the azimuthal angle with the periods contained between 2π and $2\pi/\upsilon$ and fit in the defined rule of the periods of even perturbations; see table 1 of [2]. Such state is neutrally stable against one of the odd perturbations with the period of $2\pi/\upsilon$ and stable against the other odd perturbations with the same period; if $\upsilon \geq 2$, then this state is unstable against υ-1 odd perturbations with periods exceeding $2\pi/\upsilon$ and stable against all the other odd perturbations with such periods.

4. Conclusions

A complete pattern of stability of all steady-state modes on axially symmetric cathodes is now numerically calculated.

The obtained results conform to the analytical theory [2] and also conform to the reasoning given at the end of section 4.5 of [1]. Therefore, conclusions on stability drawn in [1] remain unaltered.

5. References

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