Field to thermo-field to thermionic electron emission: a practical guide to evaluation, effect on arc cathodes

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A simple, computationally efficient, and accurate method of evaluation of field to thermo-field to thermionic electron emission current density in the framework of the Murphy-Good formalism is devised with the use of Padé approximants. It is shown that electron emission from cathodes of high-pressure arcs is adequately described by the Richardson-Schottky formula even for extremely high plasma pressures typical for some arc lamps. Emission from cathodes of vacuum arcs is of thermo-field nature and can be rather accurately described by the Hantzsche fit formula. Since no analytical formulas are uniformly valid for field to thermo-field to thermionic electron emission, a numerical evaluation of the Murphy-Good formalism is inevitable if a model is to be valid in the full range of conditions of plasma-cathode interaction in vacuum arcs. The approach proposed in this work may be the method of choice to this end.

1. Introduction

Modern multidimensional numerical models of plasma-cathode interaction in vacuum arcs require field to thermo-field to thermionic electron emission current density be evaluated at each iteration at each time step at each point of the cathode surface. Therefore, a fast and accurate evaluation method is of crucial importance. A theoretical description of field to thermo-field to thermionic electron emission from metals in the quasi-classical approximation has been developed by Murphy and Good long ago [1]. However, a question of fast and accurate evaluation in the framework of the formalism [1] is not quite trivial and has been considered in many works, e.g., [2-4]. It has become clear is that an accurate method uniformly valid from field to thermo-field to thermionic emission cannot be fully analytical.

In this work, a simple, fast, and accurate method of evaluation of field to thermo-field to thermionic electron emission current density in the framework of the formalism [1] is devised with the use of Padé approximants. Unsurprisingly, the method is not fully analytical and still involves a numerical evaluation of one integral, which is done by means of the Romberg integration. Calculations for conditions of cathodes of high-pressure and vacuum arc discharges are performed and regimes are identified where simpler descriptions are justified.

2. Evaluating the electron emission current in the framework of the Murphy-Good formalism

In the framework of the theory [1], the density of electron emission current is given by

$$j_{em}(T_w,E_w,\phi)=e\int_{-\infty}^{\infty}N(T_w,W,\phi)\,D(E_w,W)\,dW,$$

where T_w is the temperature of the surface of the emitter, E_w is the electric field at the surface, ϕ is the work function, W has the meaning of the part of the electron energy for the motion normal to the surface measured from zero for a free electron outside the metal, N is the Fermi-Dirac distribution for the free electrons in the metal,

$$N(T_w,W,\phi) = \frac{4\pi m_e k T_w}{h^3}\ln\left[1+\exp\left(-\frac{W+\phi}{kT_w}\right)\right],$$

and D is the tunnelling probability,

$$D(E_w,W) = \begin{cases} 1 & \text{for} \quad W > W_l, \\ \left\{1+\exp\left[a\frac{v(y)}{y^{3/2}}\right]\right\}^{-1} & \text{for} \quad W < W_l. \end{cases}$$

Here

$$W_l = -\sqrt{\frac{e^3E_w}{8\pi\varepsilon_0}}, \quad a = \frac{4\sqrt{2}}{3(4\pi\varepsilon_0)^{3/4}}\left(\frac{m_e^2e^5}{\hbar^4E_w}\right)^{1/4}, \quad y = \frac{\sqrt{2}\,W_l}{W},$$

and $v(y)$ is a function expressed in terms of the complete elliptic integrals $K(m)$ and $E(m)$.

Combining the above equations, one can write

$$\frac{j_{em}(T_w,E_w,\phi)}{A_{em}T_w^2} = I_1 + gI_2,$$

$$I_1 = \int_c^\infty \ln(1+e^{-z})\,dz, \qquad c = \frac{\phi+W_l}{kT_w},$$

$$I_2 = \int_{1/\sqrt{2}}^{\infty}\frac{\ln[1+\exp(gz-b)]}{1+\exp[az^{3/2}v(z^{-1})]}dz, \quad g = \frac{-\sqrt{2}W_l}{kT_w},$$

where $b=\phi/kT_w$ and $A_{em}=4\pi m_e k^2 e/h^3$.

3. Integral I_1

The function $I_1(c)$ may be expressed in terms of dilogarithm (Spence's integral for $n=2$) and the latter for $c\geq 0$ may be evaluated with the use of the

Chebyshev series. However, a faster and sufficiently accurate way is to use a Padé rational approximation (Padé approximant) over the variable $x=e^c$. The simplest rational approximation which agrees with the two-term asymptotic expansions of the function $I_1(x)$ for $x\to 0$ and $x\to\infty$ reads

$$I_1 = \frac{c_1 + c_2(x-1)}{1 + c_3(x-1) + c_2(x^2-1)}$$

with

$$c_1 = \frac{\pi^2}{12}, \quad c_2 = -\frac{1}{3}\frac{\pi^4 - 144\ln 2}{-48 + 5\pi^2}, \quad c_3 = \frac{2}{3}\frac{-6\pi^2 + \pi^4 - 54\ln 2}{-48 + 5\pi^2}.$$

The relative error of this approximant does not exceed 4.6×10^{-5} for all $c\geq 0$.

The functional relation

$$I_1(c) = \frac{\pi^2}{6} + \frac{c^2}{2} - I_1(-c)$$

will be used for $c<0$.

4. Integral I_2

This integral cannot be expressed in terms of conventional special functions and is governed by three parameters (a, g, b), so it is hardly possible to devise an accurate uniformly valid approximate formula. Therefore, the integral needs to be evaluated numerically.

A straightforward numerical evaluation of the function $v(y)$ requires an evaluation of complete elliptic integrals. The latter can be performed numerically, e.g., [5]. However, simple analytical formulas for $v(y)$ are desirable in order for numerical evaluation of the integral I_2 to be fast. Simple and accurate formulas can be derived by means of Padé approximants with the use of results [4] elucidating the character of the dependence $v(y)$ for small y. The approximant suitable in the interval $0\leq y\leq 1$ reads

$$v(y) = \frac{1-w}{1+c_4 w} + \frac{3w\ln w}{16(1+c_5 w)}$$

with $w=y^2$ and

$$c_4 = \frac{9}{8}\ln 2 - \frac{13}{16}, \quad c_5 = \frac{13 - 3c_4 - 3\pi\sqrt{2}\,(1+c_4)}{3\pi\sqrt{2}\,(1+c_4) - 16}.$$

The relative error of this approximant does not exceed 3.7×10^{-4} over the whole range $0\leq y\leq 1$, which is significantly smaller than errors of all previously reported simple formulas; an unsurprising result reflecting the power of Padé approximants. Besides, this formula ensures correct asymptotic behavior of the function $v(y)$ for both $y\to 0$ and $y\to 1$, the latter being important for deriving a smooth approximation on the whole interval $0\leq y\leq 2^{1/2}$ which is relevant for evaluation of thermo-field emission.

The approximant suitable in the interval $1\leq y\leq 2^{1/2}$ reads

$$v(y) = -\frac{3\pi}{2^{5/2}}\frac{(y-1) + c_6(y-1)^2}{1 + c_7(y-1) + c_8(y-1)^2}$$

with coefficients being expressed in terms of the functions $K(m)$ and $E(m)$ (these expressions are skipped for brevity) and having numerical values c_6=0.51470654, c_7=0.20232890, and c_8=-0.01341007. The relative error of this approximant does not exceed 4.8×10^{-6} over the whole interval $1\leq y\leq 2^{1/2}$, which again is significantly smaller than that of previously reported simple formulas.

Under conditions of practical interest, one or more parameters governing the integrand in I_2 are large and the integrand represents a multi-scale function. Therefore, an efficient numerical evaluation of integral I_2 must include an adaptive choice of the numerical grid. A suitable method is Romberg integration [5]. First, let us write

$$I_2 = \int_0^{\sqrt{2}} \frac{r_1 r_2}{y^2}\,dy,$$

where

$$r_1 = \ln\left[1 + \exp\left(\frac{g}{y} - b\right)\right], \quad r_2 = \left[1 + \exp\frac{av(y)}{y^{3/2}}\right]^{-1}.$$

In order to avoid overflow which may occur in evaluation of the exponential functions for small y, it is advisable to rewrite the last expressions as

$$r_1 = \ln\left[1 + \exp\left(b - \frac{g}{y}\right)\right] - \left(b - \frac{g}{y}\right), \quad r_2 = \frac{\exp\left[-\frac{av(y)}{y^{3/2}}\right]}{\exp\left[-\frac{av(y)}{y^{3/2}}\right] + 1}.$$

In cases where $\exp(g/y-b)$ is very small, the use of the first of these expressions causes accumulation of errors and the Romberg integration (or, more precisely, Richardson's deferred approach to the limit) may fail. Then the quantity r_1 should be evaluated by means of a series in powers of $\exp(g/y-b)$ which is obtained by expanding the logarithm.

In this framework, the Romberg integration in its standard form [5] and the above-described method on the whole are fast and robust and can be used in all conditions where the Murphy-Good theory is applicable.

5. Electron emission from cathodes of arc discharges

Let us now identify particular regimes of operation of cathodes of vacuum and high-pressure arc discharges where simpler descriptions are justified. Namely, let us consider cases where the size of non-uniformities of the cathode surface exceeds significantly the thickness of the near-cathode plasma layer and, as far as cathodes of vacuum arcs are concerned, the cathode is hot enough so that supply of cathode vapor into the discharge gap is dominated by vaporization and not explosive emission. Then the electric field at the cathode surface may be estimated by means of a 1D model of current transfer through the near-cathode plasma layer, e.g., [6,7].

Values of electric field at the cathode surface shown in Fig. 1 have been computed for a vacuum arc with a Cu cathode; for Hg and Xe high-pressure arcs for three pressure values (1, 15, and 200bar); and for a 1bar Ar arc. Note that a 1bar Ar arc represents a standard example of an atmospheric arc, while Hg and Xe arcs are of interest in connection with projection and car headlight arc lamps, where pressures of the order of 100 or 200bar are rather a rule than an exception. The near-cathode voltage drop in all the calculations reported in this work was 20V and the work function was 4.5eV. Note that the monotonically increasing dependence of E_w on T_w seen in Fig. 1 for the case of vacuum arc is a consequence of a decrease of the thickness of the space-charge sheath caused by an increase of the vapor pressure and, consequently, the local charged particle density. The switching of the dependence of $E_w(T_w)$ from increasing to decreasing in the case of high-pressure arcs occurs as the near-cathode plasma approaches full ionization.

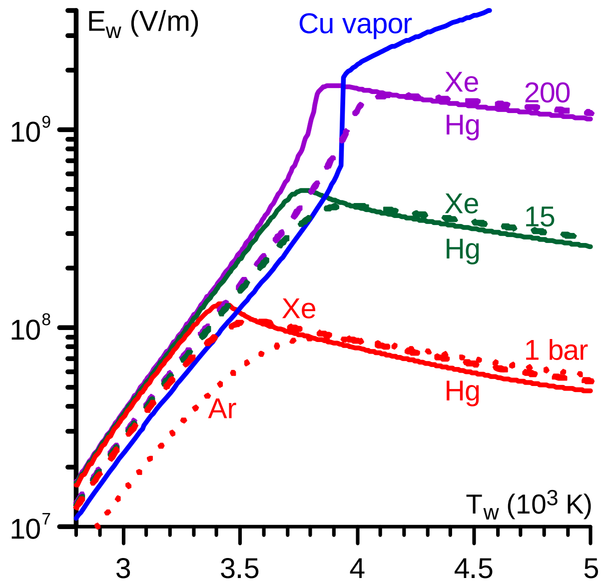

Fig. 1. Electric field at the surface of an arc cathode for different plasmas-producing gases.

Analyzing the data shown in Fig. 1 in view of those shown in Fig. 2, one concludes that the Richardson-Schottky formula represents a good approximation for the high-pressure arcs in cases p=1bar and p=15bar but may represent a poor approximation for the high-pressure arcs in the case p=200bar and for the vacuum arc. Therefore, the applicability of the Richardson-Schottky formula for the high-pressure arcs with p=200bar and for the vacuum arcs requires a more detailed investigation. In this connection, the most important parameters of the near-cathode plasma layer, which are densities of energy flux and electric current from the plasma to the cathode, evaluated using the Murphy-Good formalism and the Richardson-Schottky formula are shown in Fig. 3 for Hg and Xe arcs in the case p=200bar. One can see that the Richardson-Schottky formula represents a reasonably good approximation. The situation is different as far as the vacuum arcs are concerned, which is seen from Fig. 4: the usage of the Richardson-Schottky formula introduces a significant error.

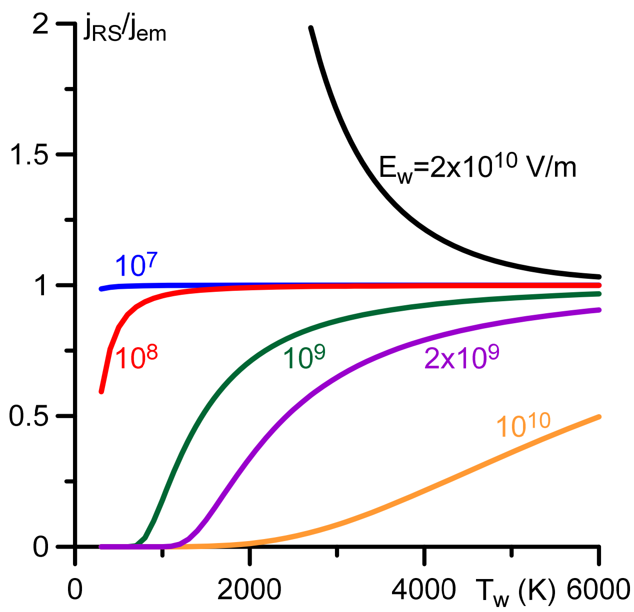

Fig. 2. Electron emission current density given by the Richardson-Schottky formula normalized by the Murphy-Good value.

Line 3 in Fig. 4 represents the density of energy flux computed using the Murphy-Good formalism with both integrals I_1 and I_2 being evaluated by means of numerical integration over the variable W with a fixed step equal to 10^{-20}J and the lower limit of integration equal to -4ϕ. There is a significant difference between this line and the line 1; an indication of importance of the use of the integration variable $1/W$ and of an adaptive choice of the numerical grid.

Line 4 in Fig. 4 represents the density of energy flux computed with the use of the Hanzsche fit formula for electron emission from a metal with the work function of 4.5eV [2] with corrections given in [8]. One can see that this formula provides a good accuracy.

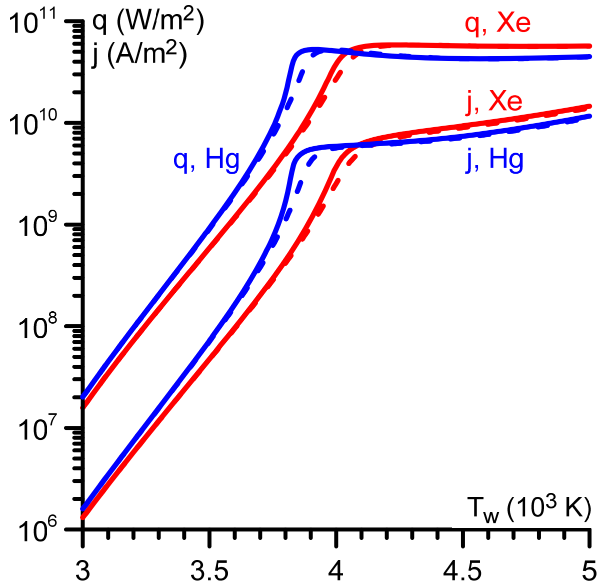

Fig. 3. Densities of energy flux and electric current from the high-pressure plasma to the cathode evaluated using the Murphy-Good formalism (solid) and the Richardson-Schottky formula (dashed).

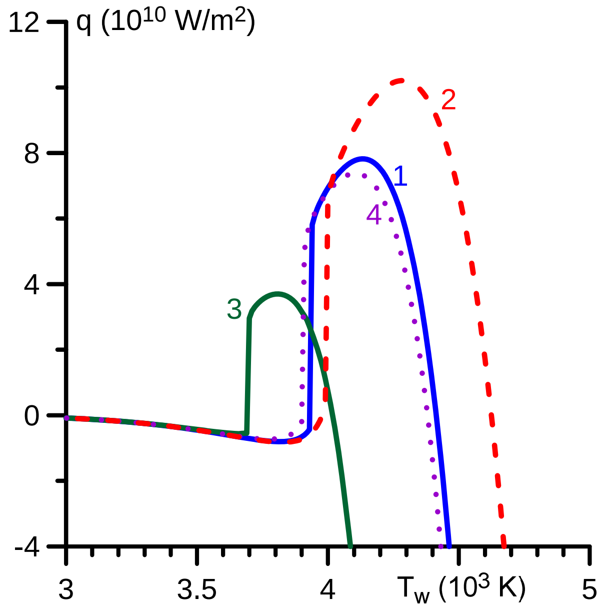

Fig. 4. Density of energy flux from the vacuum arc plasma to the Cu cathode evaluated using different descriptions of electron emission.1: the Murphy-Good formalism. 2: the Richardson-Schottky formula. 3: the Murphy-Good formalism with integration variable being the electron energy. 4: the Hantzsche formula.

5. Conclusions

A simple, accurate, and computationally efficient method of evaluation of field to thermo-field to thermionic electron emission current density in the framework of the Murphy-Good formalism is devised with the use of Padé approximants. Unsurprisingly, the method is not fully analytical and still involves a numerical evaluation of one integral. Since the integrand represents a multi-scale function, an efficient numerical evaluation of the integral must include an adaptive choice of the numerical grid. A suitable method is Romberg integration.

Calculations for conditions of cathodes of high-pressure and vacuum arc discharges are performed for cases where the size of non-uniformities of the cathode surface exceeds significantly the thickness of the near-cathode plasma layer and, as far as cathodes of vacuum arcs are concerned, the cathode is hot enough so that supply of cathode vapor into the discharge gap is dominated by vaporization and not explosive emission. It is found that electron emission from cathodes of high-pressure arcs is of thermionic nature and adequately described by the Richardson-Schottky formula even for extremely high plasma pressures (up to 200bar) typical for automotive and projection arc lamps. Emission from cathodes of vacuum arcs is of thermo-field nature and can be rather accurately described by the Hantzsche fit formula. Unfortunately, no analytical formulas are uniformly valid for field to thermo-field to thermionic electron emission, therefore a numerical evaluation of the Murphy-Good formalism is inevitable if a model is to be uniformly valid in the full range of conditions of plasma-cathode interaction in vacuum arcs. The approach proposed in this work may be the method of choice to this end.

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6. References

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