Calculation of energy fluxes from high-pressure arc plasmas to thermionic cathodes

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Abstract

This text was written in order to answer a frequently asked question of derivation of expressions given in the paper [1] for the fluxes of energy delivered to, or removed from, the cathode surface by different species and contains a detailed derivation of these expressions.

1 Kinetic energy brought to the cathode surface by ions

The distribution function of ions inside the space-charge sheath is obtained by multiplying Eq. (7) of [1] by the Maxwellian factors describing dependences of the distribution function on x- and z-components of the ion particle velocity:

$$f\left(y, v_x, v, v_z\right) = \begin{cases} \frac{1}{2\pi u_i^2} \exp\left(-\frac{v_x^2+v_z^2}{2u_i^2}\right) \frac{n_{is}}{2u_i} & \text{for } -v_+ < v < -v_- \\ 0 & \text{otherwise} \end{cases}, \tag{1}$$

where v_x, v, and v_z are the x-, y-, and z- components of the ion particle velocity.

The density of flux of kinetic energy of ions in the direction to the cathode (i.e., in the negative direction of the axis y) is

$$q_i = -\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} \frac{m_i\left(v_x^2+v^2+v_z^2\right)}{2} v f\left(y, v_x, v, v_z\right)\, dv_x\, dv\, dv_z$$

$$= -\int_{-\infty}^{\infty}\int_{-v_+}^{-v_-}\int_{-\infty}^{\infty} \frac{m_i\left(v_x^2+v^2+v_z^2\right)}{2} v \frac{1}{2\pi u_i^2} \exp\left(-\frac{v_x^2+v_z^2}{2u_i^2}\right) \frac{n_{is}}{2u_i}\, dv_x\, dv\, dv_z$$

(we introduce new integration variables $x = v_x/u_i$, $z = v_z/u_i$)

$$= -\frac{m_i n_{is} u_i}{8\pi} \int_{-\infty}^{\infty}\int_{-v_+}^{-v_-}\int_{-\infty}^{\infty} \left(x^2+z^2\right) v\, \exp\left(-\frac{x^2+z^2}{2}\right)\, dx\, dv\, dz$$

$$-\frac{m_i n_{is}}{8\pi u_i} \int_{-\infty}^{\infty}\int_{-v_+}^{-v_-}\int_{-\infty}^{\infty} v^3 \exp\left(-\frac{x^2+z^2}{2}\right)\, dx\, dv\, dz$$

(note that the first term describes the flux of kinetic energy of motion in the x and z-directions while the second term describes the flux of kinetic energy of motion in the y-direction)

$$= -\frac{m_i n_{is} u_i}{8\pi} \underbrace{\iint\limits_{-\infty}^{\infty} \left(x^2+z^2\right) \exp\left(-\frac{x^2+z^2}{2}\right) dx\, dz}_{4\pi} \underbrace{\int_{-v_+}^{-v_-} v\, dv}_{\frac{v_-^2 - v_+^2}{2}}$$

$$-\frac{m_i n_{is}}{8\pi u_i} \underbrace{\iint\limits_{-\infty}^{\infty} \exp\left(-\frac{x^2+z^2}{2}\right) dx\, dz}_{2\pi} \underbrace{\int_{-v_+}^{-v_-} v^3\, dv}_{\frac{v_-^4 - v_+^4}{4}}$$

$$= -\frac{m_i n_{is} u_i}{8\pi} 4\pi \frac{v_-^2 - v_+^2}{2} - \frac{m_i n_{is}}{8\pi u_i} 2\pi \frac{v_-^4 - v_+^4}{4} = -m_i n_{is} u_i \frac{v_-^2 - v_+^2}{4}\left(1 + \frac{v_-^2 + v_+^2}{4u_i^2}\right)$$

[we make use of Eq. (8) of [1]]

$$= -m_i n_{is} u_i \frac{(v_s - u_i)^2 - (v_s + u_i)^2}{4}\left[1 + \frac{(v_s - u_i)^2 + (v_s + u_i)^2 - \frac{4Ze\varphi}{m_i}}{4u_i^2}\right] =$$

$$= m_i n_{is} u_i^2 v_s \left(1 + \frac{m_i v_s^2 + m_i u_i^2 - 2Ze\varphi}{2m_i u_i^2}\right) = \underbrace{n_{is} v_s}_{J_i} \left(\underbrace{m_i u_i^2}_{kT_i} + \underbrace{\frac{m_i v_s^2}{2}}_{\frac{kT_i + ZT_e}{2}} + \underbrace{\frac{m_i u_i^2}{2}}_{\frac{kT_i}{2}} - Ze\varphi\right) =$$

$$= J_i \left(kT_i + kT_i + \frac{ZT_e}{2} - Ze\varphi\right). \tag{2}$$

On the cathode surface $\varphi = -U_D$ and the energy flux to the cathode is

$$q_i = J_i \left(2kT_i + \frac{ZT_e}{2} + ZeU_D\right),$$

in accord to what is stated in the first paragraph on page 1874 of [1]. One can see from the derivation [in particular, from Eq. (2)] that the flux of kinetic energy of motion of ions in the x and z-directions is $J_i kT_i$. This is what one would intuitively expect taking into account that the distribution in these directions is Maxwellian and the average kinetic energy of chaotic motion corresponding to each degree of freedom is $\frac{kT_i}{2}$. However, the distribution in the y-direction is not Maxwellian, i.e., motion of ions in the y-direction is not chaotic, hence one should not expect the flux of kinetic energy of motion of ions in the y-direction to be equal to $J_i \frac{kT_i}{2}$. Indeed, one can see from Eq. (2) that this flux equals $J_i \left(kT_i + \frac{ZT_e}{2} + ZeU_D\right)$, i.e., is very different from $J_i \frac{kT_i}{2}$.

In fact, there is one more reason to expect the flux of kinetic energy of motion of ions in the y-direction not to be equal to $J_i \frac{kT_i}{2}$. This flux is obtained by averaging the product of the ion kinetic energy $\frac{m_i v^2}{2}$ times the y-component v of the ion particle velocity and this average value is *not* equal to the product of average values (which would give $\frac{kT_i}{2}J_i$ in the case of Maxwellian distribution). Hence, the flux of kinetic energy of motion of ions in the y-direction is different from $J_i\frac{kT_i}{2}$ even if the ion distribution were Maxwellian. This conclusion will be illustrated by the calculation of the flux of kinetic energy of plasma electrons given in the next section.

2 The kinetic energy brought to the cathode surface by plasma electrons

The distribution function of plasma electrons inside the space-charge sheath is Maxwellian:

$$f(y, v_x, v, v_z) = \frac{n_e}{(2\pi)^{3/2} u_e^3} \exp\left(-\frac{v_x^2 + v^2 + v_z^2}{2u_e^2}\right),$$

where v_x, v, and v_z are the x-, y-, and z- components of the electron particle velocity, $u_e = \sqrt{kT_e/m_e}$, and the electron number density n_e is given by Eq. (4) of [1]. The number density of flux of electrons to the cathode surface is

$$J_e = -\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} v f(y, v_x, v, v_z)\ dv_x\ dv\ dv_z$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} v\frac{n_{ew}}{(2\pi)^{3/2} u_e^3} \exp\left(-\frac{v_x^2 + v^2 + v_z^2}{2u_e^2}\right)\ dv_x\ dv\ dv_z,$$

where n_{ew} is the electron number density at the cathode surface. One finds

$$J_e = n_{ew}\underbrace{\frac{1}{2\pi u_e^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{v_x^2 + v_z^2}{2u_e^2}\right)\ dv_x\ dv_z}_{1}\underbrace{\frac{1}{(2\pi)^{1/2} u_e}\int_{0}^{\infty} v \exp\left(-\frac{v^2}{2u_e^2}\right)\ dv}_{\frac{u_e}{\sqrt{2\pi}}}.$$

Thus,

$$J_e = \frac{n_{ew}u_e}{\sqrt{2\pi}} \tag{3}$$

or, equivalently, $J_e = \frac{1}{4}n_{ew}\overline{C}_e$, where $\overline{C}_e = (8kT_e/\pi m_e)^{1/2}$ is the electron mean chaotic speed; a well-known result available in textbooks. Since $n_{ew} = n_{es}\exp\left(-\frac{eU_D}{kT_e}\right)$, Eq. (3) coincides with Eq. (1) of [1].

The density of flux of kinetic energy of electrons to the cathode is

$$q_e = -\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} \frac{m_e\left(v_x^2 + v^2 + v_z^2\right)}{2} v f(y, v_x, v, v_z)\ dv_x\ dv\ dv_z \tag{4}$$

$$= -\frac{n_{ew}m_e}{2(2\pi)^{3/2} u_e^3}\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} \left(v_x^2 + v_z^2\right) v \exp\left(-\frac{v_x^2 + v^2 + v_z^2}{2u_e^2}\right)\ dv_x\ dv\ dv_z$$

$$-\frac{n_{ew}m_e}{2(2\pi)^{3/2} u_e^3}\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} v^3 \exp\left(-\frac{v_x^2 + v^2 + v_z^2}{2u_e^2}\right)\ dv_x\ dv\ dv_z \tag{5}$$

$$= -\frac{n_{ew} m_e u_e^3}{2\,(2\pi)^{3/2}} \underbrace{\frac{1}{u_e^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(v_x^2 + v_z^2\right) \exp\left(-\frac{v_x^2 + v_z^2}{2u_e^2}\right) \, dv_x \, dv_z}_{4\pi} \underbrace{\frac{1}{u_e^2} \int_{-\infty}^{0} v \exp\left(-\frac{v^2}{2u_e^2}\right) \, dv}_{-1}$$

$$-\frac{n_{ew} m_e u_e^3}{2\,(2\pi)^{3/2}} \underbrace{\frac{1}{u_e^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{v_x^2 + v_z^2}{2u_e^2}\right) \, dv_x \, dv_z}_{2\pi} \underbrace{\frac{1}{u_e^4} \int_{-\infty}^{0} v^3 \exp\left(-\frac{v^2}{2u_e^2}\right) \, dv}_{-2} \qquad (6)$$

$$= J_e k T_e + J_e k T_e \qquad (7)$$

$$= 2 J_e k T_e$$

in accord to what is stated in the third paragraph on the page 1874 [1]. One can see from the derivation [in particular, from Eq. (7)] that the flux of kinetic energy of motion of electrons in the x and z-directions is $J_e k T_e$ while the flux of kinetic energy of motion of electrons in the y-direction is $J_e k T_e$ rather than $J_e \frac{kT_e}{2}$. As explained at the end of the previous section, this is due to the fact that the latter flux equals the average value of the product of kinetic energy of motion of electrons in the y-direction times the y-component v of the electron particle velocity, and this average value is *not* equal to the product of average values. The latter is clearly seen from the evaluation of the second term in Eq. (6).

Expressions for the fluxes of kinetic energy of the emitted electrons and neutral atoms leaving the cathode surface, given in the fourth paragraph on the page 1874 [1], are derived in the same way.

References

[1] M. S. Benilov and A. Marotta, J. Phys. D: Appl. Phys. **28**, 1869 (1995).