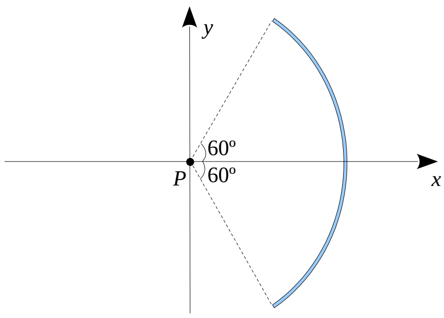
UNIVERSIDADE da MADEIRA

Electromagnetismo

Series of problems 2

1. The figure shows a plastic rod that has a uniformly distributed charge $-Q$. The rod was bent in a circular arc of $120\,°$ and radius r. The axis of coordinates is chosen in a way that the origin of the axis lies at the center of curvature P of the rod. What is the electric field $\mathbf{E}$ due to the rod at point P, in terms of Q and r?



$\boxed{\text{Solution 1:}}$ The electric field may be written as: $\mathbf{E} = \frac{Kq}{r^2}\widehat{u}_r$

For a differential element of the rod we have:: $d\mathbf{E} = \frac{Kdq}{r^2}\widehat{u}_r$

$\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2 = -\mathbf{r}_2 \longrightarrow \widehat{r}_{12} = -\frac{\mathbf{r}_2}{r_2}$

$$\begin{cases} \mathbf{r}_2 = (r_{2x}; r_{2y}) \\ \sin\theta = \frac{r_{2y}}{r_2}; \cos\theta = \frac{r_{2x}}{r_2} \end{cases} \longrightarrow \widehat{r}_{12} = -\frac{\mathbf{r}_2}{r_2} = -\frac{r_2(\cos\theta;\sin\theta)}{r_2} = -\left(\cos\theta;\sin\theta\right)$$

We can write: $d\mathbf{E} = \frac{Kdq}{r^2}\widehat{u}_r = -\frac{Kdq}{r^2}\left(\cos\theta;\sin\theta\right)$

$\mathbf{E} = \int d\mathbf{E} = -\int \frac{Kdq}{r^2}\left(\cos\theta;\sin\theta\right)$

Since $\begin{cases} \frac{dq}{dl} = \lambda \\ dl = rd\theta \end{cases},$

$$\mathbf{E} = -\int_{\theta=-60\,°}^{\theta=60\,°} \frac{K\lambda r d\theta}{r^2}\left(\cos\theta;\sin\theta\right) = -\frac{K\lambda}{r}\int_{\theta=-60\,°}^{\theta=60\,°}\left(\cos\theta;\sin\theta\right)d\theta$$
$$= -\frac{K\lambda}{r}\left(\int_{\theta=-60\,°}^{\theta=60\,°}\cos\theta d\theta; \int_{\theta=-60\,°}^{\theta=60\,°}\sin\theta d\theta\right) = -\frac{K\lambda}{r}\left([\sin\theta]_{-60\,°}^{+60\,°}; [-\cos\theta]_{-60\,°}^{+60\,°}\right) = -\frac{K\lambda}{r}\left(2\sin 60\,°;0\right)$$

Let's now find the value of λ: $\lambda = \frac{q}{\text{length}} = \frac{-Q}{\frac{2\pi r}{3}} = \frac{-3Q}{2\pi r}$.

$$\mathbf{E} = -\frac{K\lambda}{r}\left(2\sin 60\,°;0\right) = -\frac{K\left(\frac{-3Q}{2\pi r}\right)}{r}\left(2\sin 60\,°;0\right) = \frac{3QK}{\pi r^2}\left(\sin 60\,°;0\right) = \frac{3Q}{4\varepsilon_0\pi^2 r^2}\left(\sin 60\,°;0\right)\ \mathrm{N\,C^{-1}}.$$

$\boxed{\text{Solution 2:}}$ As can be seen from the figure, y components cancel each other due to symmetry. Therefore $E_P = \int dE_x$.

$$\begin{cases} dE = K\frac{dq}{r^2} \\ \frac{dq}{ds} = \lambda \\ dE_x = \cos\theta dE \end{cases} \Longrightarrow E_P = \int dE_x = \int \cos\theta dE = \int \cos\theta K\frac{dq}{r^2} = \int \cos\theta K\frac{\lambda ds}{r^2}$$

There are two variables. Before proceeding with the integration we must eliminate one of them:

$ds = rd\theta$.

$E_P = \int \cos\theta K\frac{\lambda ds}{r^2} = \int_{\theta=-60^\circ}^{\theta=+60^\circ} K\frac{\lambda r\cos\theta d\theta}{r^2} = \frac{K\lambda}{r}\int_{\theta=-60^\circ}^{\theta=+60^\circ}\cos\theta d\theta = \frac{K\lambda}{r}\left[\sin\theta\right]_{-60^\circ}^{+60^\circ}$

$= \frac{K\lambda}{r}\left[\sin 60^\circ - \sin(-60^\circ)\right] = \frac{K\lambda}{r}\left[\sin 60^\circ + \sin(60^\circ)\right] = 2\frac{K\lambda}{r}\sin 60^\circ = 1.732\frac{K\lambda}{r}$.

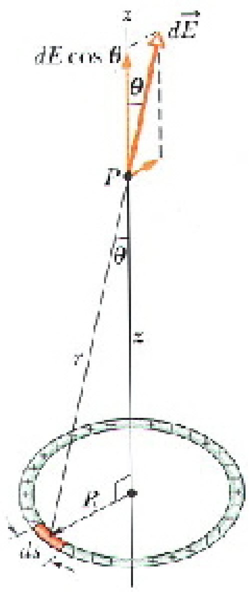
E.

Let's find the value of λ: $\lambda = \frac{q}{\text{length}} = \frac{Q}{\frac{2\pi r}{3}} = \frac{3Q}{2\pi r} = 0.477\frac{Q}{r}$.

Finally: $E_P = 1.732\frac{K\lambda}{r} = 1.732\frac{K}{r}\left(0.477\frac{Q}{r}\right) = \frac{0.83KQ}{r^2}$

In vector form: $\mathbf{E}_P = \frac{0.83KQ}{r^2}\mathbf{u}_x$ where Q is positive.

2. Calculate the electric field at point P caused by a ring of very thin thickness along the central axis of the ring at a distance z from the plane of the ring along the axis. The radius of the ring is R and has a uniform and positive linear charge density (■).



$\boxed{\text{Solution:}}$ The electric field dE always has two components: one parallel to the z axis and one perpendicular. The components parallel to the z axis all have the same direction, and the same magnitude. The components perpendicular to the axis eventually cancel each other out.

$E_P = \int dE_z$.

$$\begin{cases} dE = K\frac{dq}{r^2} \\ \frac{dq}{ds} = \lambda \\ dE_z = \cos\theta dE \end{cases} \Longrightarrow E_P = \int dE_z = \int \cos\theta dE = \int \cos\theta K\frac{dq}{r^2} = \int \cos\theta K\frac{\lambda ds}{r^2} = K\lambda\int\cos\theta\frac{ds}{r^2}$$

Since $\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{R^2+z^2}}$.

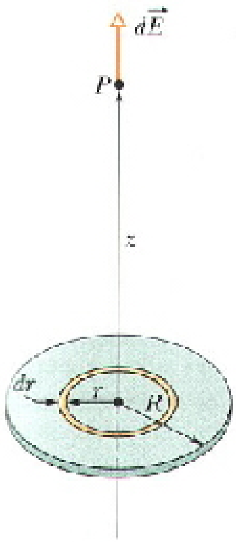
$E_P = K\lambda \int \cos\theta \frac{ds}{r^2} = K\lambda \int \frac{z}{\sqrt{R^2+z^2}} \frac{ds}{R^2+z^2} = \frac{K\lambda z}{(R^2+z^2)^{\frac{3}{2}}} \int_{s=0}^{s=2\pi R} ds = \frac{K\lambda z}{(R^2+z^2)^{\frac{3}{2}}} (2\pi R) = \frac{2\pi R K \lambda z}{(R^2+z^2)^{\frac{3}{2}}}$

Let's find the value of λ: $\lambda = \frac{Q}{\text{length}} = \frac{q}{2\pi R}$.

Finally: $E_P = \frac{2\pi R K \frac{q}{2\pi R} z}{(R^2+z^2)^{\frac{3}{2}}} = \frac{Kqz}{(R^2+z^2)^{\frac{3}{2}}}$.

Note: For large distances $z \gg R \Longrightarrow E_P \approx \frac{Kq}{z^2}\,\mathrm{N\,C^{-1}}$, that is, it is as if it were a point load that was producing the field.

3. The figure shows a plastic disc with radius R having a positive surface charge density ■ uniformly on the top surface.



(a) What is the electric field at point P at the distance z from the disc along the central axis?

(b) Derive from the expression found in the previous paragraph the expression for very large distances z.

$\boxed{\text{Solution:}}$ a) Using the previous exercise we know that the field at point P due to a very thin ring is given by: $\frac{Kzdq}{(r^2+z^2)^{\frac{3}{2}}}$.

If we add the contribution made by all the rings we have the total field: $E_P = \int \frac{Kzdq}{(r^2+z^2)^{\frac{3}{2}}}$.

dq relates to surface charge density as follows: $\sigma = \frac{dq}{dA}$. dA can be written as: $dA = 2\pi r dr$.

So we can write: $\sigma = \frac{dq}{2\pi r dr}$

$E_P = \int \frac{Kzdq}{(r^2+z^2)^{\frac{3}{2}}} = \int \frac{Kz\sigma 2\pi r dr}{(r^2+z^2)^{\frac{3}{2}}} = \pi\sigma z K \int_{r=0}^{r=R} \frac{2rdr}{(r^2+z^2)^{\frac{3}{2}}}$

To solve this integral it is worth remembering the following formula:

$\int y^m dy = \frac{y^{m+1}}{m+1}$; be $y = r^2 + z^2$, $m = -\frac{3}{2}$, $\frac{dy}{dr} = 2r \longrightarrow dy = 2rdr$

So we have: $\int (r^2+z^2)^{-\frac{3}{2}} 2rdr = \int y^m dy = \frac{y^{m+1}}{m+1} = \frac{(r^2+z^2)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} = -\frac{2}{\sqrt{r^2+z^2}}$

$E_P = \pi\sigma z K \left[-\frac{2}{\sqrt{r^2+z^2}}\right]_{r=0}^{r=R} = \pi\sigma z K \left[-\frac{2}{\sqrt{R^2+z^2}} + \frac{2}{z}\right]$

$= 2\pi\sigma K \left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\,\mathrm{N\,C^{-1}}$.

Another possible answer: $E_P = 2\pi\sigma\frac{1}{4\pi\varepsilon_0}\left[1-\frac{z}{\sqrt{R^2+z^2}}\right] = \frac{\sigma}{2\varepsilon_0}\left[1-\frac{z}{\sqrt{R^2+z^2}}\right]\,\mathrm{N\,C^{-1}}$.

b) If we replace R^2+z^2 by z^2 (when $z \gg R$) we find $E \longrightarrow 0$. Although it is correct, we are interested in the dependency, and this tells us nothing about it. We can find the desired dependency using binomial expansion $(1+\varepsilon)^n \approx 1+n\varepsilon$, when $\varepsilon \to 0$. We then have: $\frac{z}{\sqrt{R^2+z^2}} = \frac{1}{\left(\left(\frac{R}{z}\right)^2+1\right)^{\frac{1}{2}}} = \left(1+\left(\frac{R}{z}\right)^2\right)^{-\frac{1}{2}} \approx 1-\frac{1}{2}\left(\frac{R}{z}\right)^2$.

Now if we use only the first term of this expansion we get the result already discussed ($E=0$), but if we use the first two terms we get

$E_P = \frac{\sigma}{2\varepsilon_0}\left[1-\frac{z}{\sqrt{R^2+z^2}}\right] \approx \frac{\sigma}{2\varepsilon_0}\left[1-\left(1-\frac{1}{2}\left(\frac{R}{z}\right)^2\right)\right] = \frac{\sigma}{2\varepsilon_0}\frac{1}{2}\left(\frac{R}{z}\right)^2$

Since $\sigma = \frac{Q}{\pi R^2}$ we get: $E_P = \frac{\sigma}{2\varepsilon_0}\frac{1}{2}\left(\frac{R}{z}\right)^2 = \frac{Q}{2\varepsilon_0\pi R^2}\frac{1}{2}\left(\frac{R}{z}\right)^2 = \frac{Q}{4\pi\varepsilon_0 z^2} = K\frac{Q}{z^2}\,\mathrm{N\,C^{-1}}$.

Or using the other expression: $E_P = 2\pi\sigma K\left[1-\frac{z}{\sqrt{R^2+z^2}}\right] \approx 2\pi\sigma K\left[1-\left(1-\frac{1}{2}\left(\frac{R}{z}\right)^2\right)\right] = K\pi\sigma\left(\frac{R}{z}\right)^2$, since $\sigma = \frac{Q}{\pi R^2}$ we get: $E_P = K\pi\frac{Q}{\pi R^2}\left(\frac{R}{z}\right)^2 = K\frac{Q}{z^2}\,\mathrm{N\,C^{-1}}$.

4. An electric field is given by $\mathbf{E} = \left(\frac{x}{2}+2y\right)\mathbf{a}_x + 2x\mathbf{a}_y$ ($\mathrm{V\,m^{-1}}$). Find the work done in moving a point load $Q = -20\,\mu\mathrm{C}$:

   (a) from origin to$(4,0,0)$ m.

   (b) from $(4,0,0)$ m to $(4,2,0)$ m.

5. For the electric field from the previous exercise, find the work done to move the same charge from $(4,2,0)$ m to the origin along a straight line.

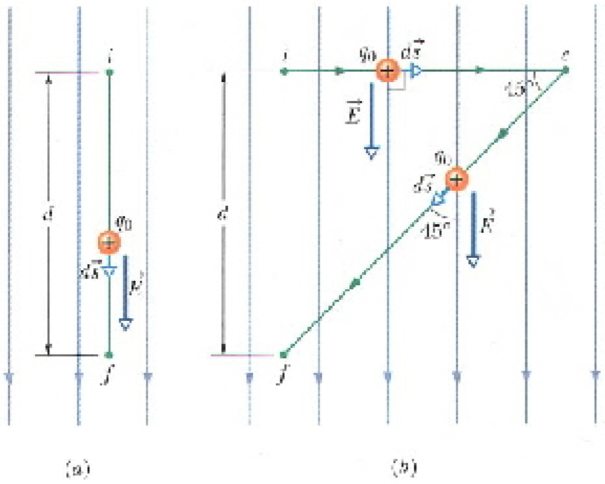
6. An electric field is given by $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z$. Determine the work carried out in transporting a $2\,\mathrm{C}$ charge from $B\,(1,0,1)$ to $A\,(0.8,0.6,1)$ along the shortest arc of the circle $x^2+y^2=1; z=1$.

7. Determine again the work to relocate $2\,\mathrm{C}$ from B to A in the same electric field as the previous exercise, but this time through a straight line between B and A.

8. Electrons are continuously ejected from air molecules in the atmosphere by the cosmic ray particles from space. Once released, each electron is subjected to an electrostatic force $\mathbf{F}$ due to the electric field $\mathbf{E}$ which is produced in the atmosphere by charged particles already existing on Earth. Near the earth's surface the electric field has magnitude $E = 150\,\mathrm{N\,C^{-1}}$ and is directed downward. What is the variation in the electric potential energy ■U of a dropped electron when the electrostatic force causes it to move vertically upwards a distance $d = 520\,\mathrm{m}$? Assume that the electron charge is $-1.6\times10^{-19}\,\mathrm{C}$.

9. Figure a) shows two points i and f in a uniform electric field $\mathbf{E}$. The points are on the same

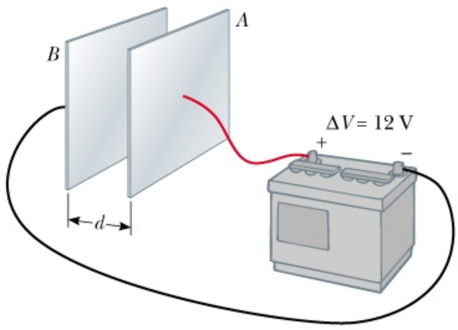
electric field line (not shown) and are separated by a distance d.



(a) Find the potential difference $\varphi_f - \varphi_i$ by moving a positive probing charge q_0 from i to f along the path shown in figure (a), which is paralell to the direction of the field.

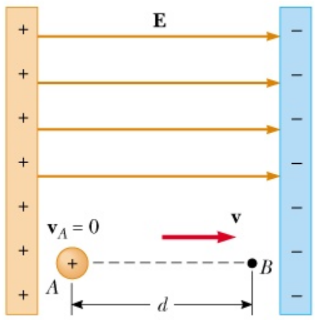
(b) Find the potential difference $\varphi_f - \varphi_i$ by moving a positive probing charge from i to f along the icf path shown in figure (b).

10. A battery has its terminals connected to two parallel cards, as shown in the figure. The potential difference of the battery is $12\,\mathrm{V}$. The separation between the plates is $d = 0.30\,\mathrm{cm}$, and it is assumed that the electric field between the plates is uniform. (This assumption is reasonable if the separation between the plates is small relative to the plate dimensions and if we do not consider positions near the edges of the plates.) Find the magnitude of the electric field between the plates.



11. A proton is dropped from rest in a uniform electric field that has a magnitude of $8,0 \times 10^4\,\mathrm{V\,m^{-1}}$ (see figure). The proton is displaced $0.50\,\mathrm{m}$ in the direction of $\mathbf{E}$. Consider that the proton charge

is 1.60×10^{-19} C and its mass 1.67×10^{-27} kg.



Determine:

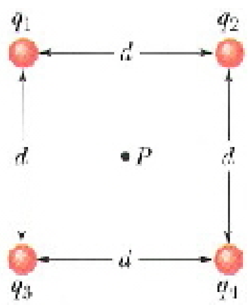
(a) The potential difference between points A e B ($V_B - V_A$).

(b) The potential energy variation of the proton-field system for this displacement.

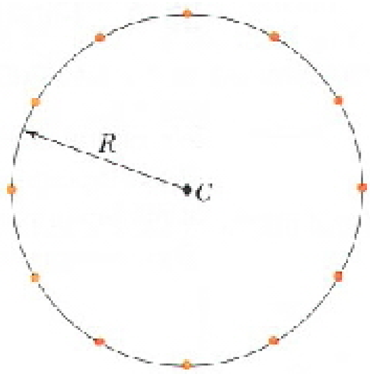
(c) proton speed after completing the displacement.

12. What is the electrical potential at point P, located in the center of the square of point charges shown in the figure? The distance d is 1.3 m and the charges are

$$\begin{aligned} q_1 &= +12\,\text{nC} \qquad q_3 = +31\,\text{nC} \\ q_2 &= -24\,\text{nC} \qquad q_4 = +17\,\text{nC} \end{aligned}$$



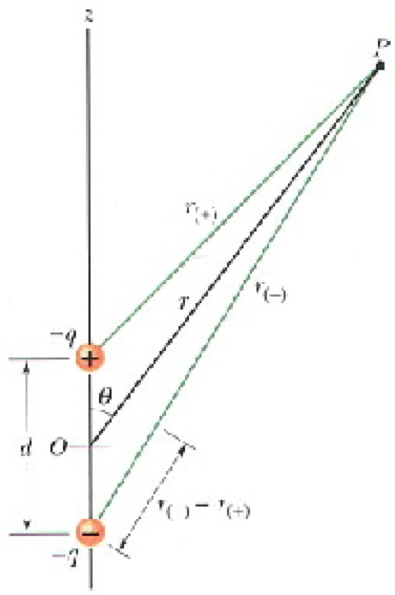
13. In the figure, 12 electrons are equally spaced and fixed around a circle of radius R. Relative to a potential that is zero at infinite, what are the electric potential and the electric field in the center of the circle due to these electrons?



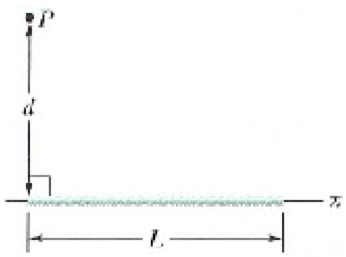
14. Consider the electric dipole in the figure.

    (a) Find the potencial φ at an arbitrary point P.

    (b) Find and approximate expression for the particular case $r \gg d$.

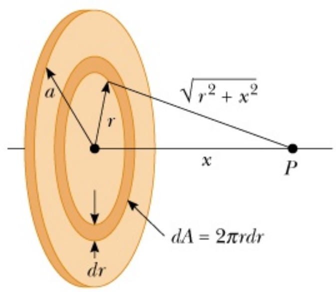


15. In the figure a thin nonconducting rod of length L has a positive and uniform charge of linear density λ. Determine the potential of the electric field induced by the rod at point P which is at a distance d perpendicular to the left end of the rod.



Reference information: $\int \frac{dz}{(z^2+a^2)^{\frac{1}{2}}} = \ln\left[z + (z^2 + a^2)^{\frac{1}{2}}\right]$

16. The figure shows a plastic disc with radius a having a positive surface charge of uniform density σ at the top of the surface. Find an expression for $\varphi(x)$, the electrical potential at any point on the central axis.



17. The electrical potential at any point on the central axis of a uniformly charged disc is (refer to

the figure of the previous problem):

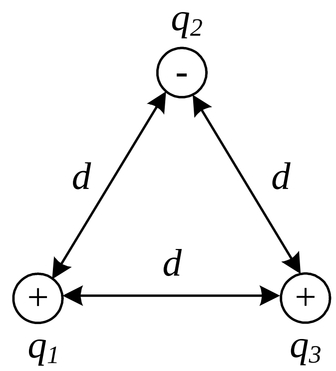
$$\varphi = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{x^2+a^2} - x\right).$$

Using this expression find an expression for the electric field at any point on the disk central axis.

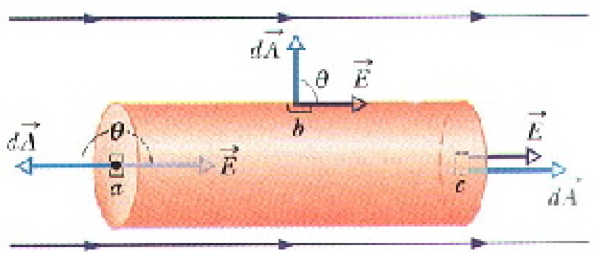
18. The figure shows three point loads held in fixed positions by forces not shown. What is the potential electrical energy U of this charge system? Assume that $d = 12\,\mathrm{cm}$ and

$$q_1 = +q \qquad q_2 = -4q \qquad q_3 = +2q,$$

where $q = 150\,\mathrm{nC}$.



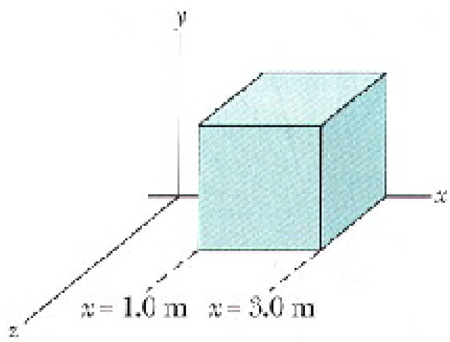
19. The figure shows a Gaussian surface in the form of a cylinder with radius R immersed in a uniform electric field $\mathbf{E}$, with the cylinder axis parallel to the field. What is the flux Φ of the electric field through this closed surface?



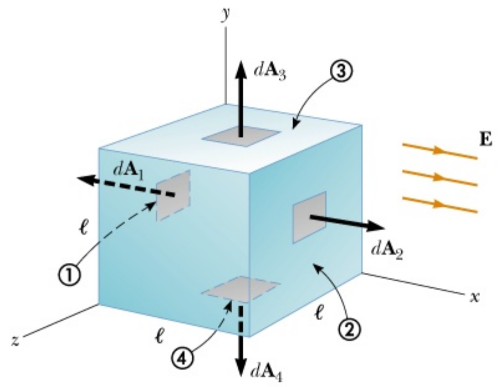
20. What is the electrical flux through a sphere that has a radius $r = 1.00\,\mathrm{m}$ and possesses a charge $1.00\,\mu\mathrm{C}$ at its centre?

21. A non-uniform electric field given by $\mathbf{E} = 3.0x\hat{\imath} + 4.0\hat{\jmath}$ pierces the Gaussian cube illustrated in the figure. (E is in units of newtons per coulomb and x in meters.) What is the electric flux through

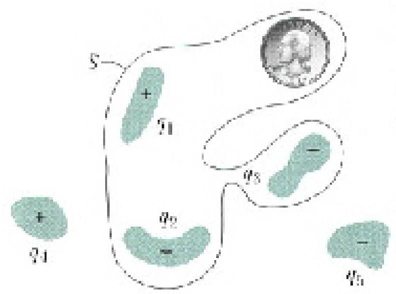
the right face, the left face, and the top face?



22. Consider a uniform electric field **E** parallel to the x axis. Find the total electric flux across the surface of a cube with side l oriented as shown in the figure.



23. The figure shows five charged plastic pieces and one electrically neutral coin. A Gaussian surface is indicated (S). What is the total electric flux across the surface if $q_1 = q_4 = +3,1\,\mathrm{nC}$, $q_2 = q_5 = -5,9\,\mathrm{nC}$, and $q_3 = -3,1\,\mathrm{nC}$?



24. From Gauss's law, calculate the electric field due to an isolated point charge q.

25. An isolated solid sphere of radius a has a uniform density ρ and carries a positive total charge Q. Calculate the magnitude of the electric field in a point:

   (a) exterior to the sphere;

   (b) interior to the sphere.

26. A thin spherical surface of radius a has a total charge Q evenly distributed on its surface. Find the electric field in points:

(a) on the surface;

(b) inside the surface.

27. Consider an infinitely long line of positive charge and with constant charge density per unit length equal to λ. Find the electric field, caused by the line, at a distance r perpendicular to the line.

28. Find the electric field due to an infinite plane with positive charge with uniform surface charge density σ.

Soluções:

1) $\mathbf{E}_P = \frac{3QK}{\pi r^2}(\sin 60\,°; 0) = \frac{3Q}{4\varepsilon_0 \pi^2 r^2}(\sin 60\,°; 0) = \frac{1}{4\pi\varepsilon_0}\frac{0.83Q}{r^2}\mathbf{u}_x \quad (\mathrm{N\,C^{-1}})$; 2) $E_P = \frac{1}{4\pi\varepsilon_0}\frac{2\pi R\lambda z}{(R^2+z^2)^{\frac{3}{2}}} \quad (\mathrm{N\,C^{-1}})$; 3a) $E_P = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right) \quad (\mathrm{N\,C^{-1}})$; 3b) $E_P = \frac{1}{4\varepsilon_0}\frac{\sigma R^2}{z^2}$; 4a) $80\,\mu\mathrm{J}$; 4b) $320\,\mu\mathrm{J}$; 5) $-400\,\mu\mathrm{J}$; 6) $-0,96\,\mathrm{J}$;7) $-0,96\,\mathrm{J}$; 8) $\Delta U = -1,2\times 10^{-14}\,\mathrm{J}$; 9a) $\varphi_f - \varphi_i = -Ed$; 9b) $\varphi_f - \varphi_i = -Ed$; 10) $E = 4,0\times 10^3\,\mathrm{V\,m^{-1}}$; 11a) $\Delta\varphi = -4,0\times 10^4\,\mathrm{V}$; 11b) $\Delta U = -6,4\times 10^{-15}\,\mathrm{J}$; 11c) $v = 2,8\times 10^6\,\mathrm{m\,s^{-1}}$; 12) $\varphi \approx 350\,\mathrm{V}$; 13) $\varphi = -\frac{12}{4\pi\varepsilon_0}\frac{e}{R}$, $\mathbf{E} = 0$; 14a) $\varphi = \frac{q}{4\pi\varepsilon_0}\frac{r_{(-)}-r_{(+)}}{r_{(-)}r_{(+)}}$; 14b) $\varphi = \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2}$; 15) $\varphi = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{L+\left(L^2+d^2\right)^{\frac{1}{2}}}{d}\right]$; 16) $\varphi(x) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{x^2+a^2} - x\right)$; 17) $E_x = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{x}{\sqrt{x^2+a^2}}\right)$; 18) $U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{r_{12}} + \frac{q_2q_3}{r_{23}} + \frac{q_1q_3}{r_{13}}\right)$; 19) $\Phi = 0\,\mathrm{N\,m^2\,C^{-1}}$; 20) $\Phi = 1,13\times 10^5\,\mathrm{N\,m^2\,C^{-1}}$; 21) $\Phi_d = 36\,\mathrm{N\,m^2\,C^{-1}}, \Phi_e = -12\,\mathrm{N\,m^2\,C^{-1}}, \Phi_t = 16\,\mathrm{N\,m^2\,C^{-1}}$; 22) $\Phi_t = 0\,\mathrm{N\,m^2\,C^{-1}}$; 23) $\Phi = -670\,\mathrm{N\,m^2\,C^{-1}}$; 24) $E = k_e\frac{q}{r^2}$; 25a) $E = k_e\frac{q}{r^2}$; 25b) $E = k_e\frac{q}{a^3}r$; 26a) $E = k_e\frac{q}{r^2}$; 26b) $E = 0$; 27) $E = 2k_e\frac{\lambda}{r}$; 28) $E = \frac{\sigma}{2\varepsilon_0}$